SUPERHIGHNESS

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ABSTRACT. We prove that superhigh sets can be jump traceable, answering a question of Cole and Simpson. On the other hand, we show that such sets cannot be weakly 2-random. We also study the class superhigh$^\diamond$, and show that it contains some, but not all, of the noncomputable K-trivial sets.

1. INTRODUCTION

An important non-computable set of integers in computability theory is $\emptyset'$, the halting problem for Turing machines. Over the last half century many interesting results have been obtained about ways in which a problem can be almost as hard as $\emptyset'$. The *superhigh* sets are the sets A such that

$$A' \geq_{tt} \emptyset'',$$

i.e., the halting problem relative to A computes $\emptyset''$ using a truth-table reduction. The name comes from comparison with the *high* sets, where instead arbitrary Turing reductions are allowed ($A' \geq_T \emptyset''$). Superhighness for computably enumerable (c.e.) sets was introduced by Mohrherr [M]. She proved that the superhigh c.e. degrees sit properly between the high and Turing complete ($A \geq_T \emptyset'$) ones.

Most questions one can ask on superhighness are currently open. For instance, Martin [M] (1966) famously proved that a degree is high iff it can compute a function dominating all computable functions, but it is not known whether superhighness can be characterized in terms of domination. Cooper [C] showed that there is a high minimal Turing degree, but we do not know whether a superhigh set can be of minimal Turing degree. We hope the present paper lays the groundwork for a future understanding of these problems.

We prove that a superhigh set can be jump traceable. Let superhigh$^\diamond$ be the class of c.e. sets Turing below all Martin-Löf random (ML-random) superhigh sets (see [N, Section 8.5]). We show that this class contains a promptly simple set, and is a proper subclass of the c.e. K-trivial sets. This class was recently shown to coincide with the strongly jump traceable c.e. sets, improving our result [A].

Definition 1.1. *Let $\{\Phi_n^X\}_{n\in\mathbb{N}}$ denote a standard list of all functions partial computable in X, and let W_n^X denote the domain of Φ_n^X. We write $J^X(n)$ for $\Phi_n^X(n)$, and $J^\sigma(n)$ for $\Phi_n^\sigma(n)$ where σ is a string. Thus $X' = \{e\colon\ J^B(e)\downarrow\}$ represents the halting problem relative to X.*

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X is jump-traceable by Y *(written $X \leq_{JT} Y$) if there exist computable functions $f(n)$ and $g(n)$ such that for all n, if $J^X(n)$ is defined ($J^X(n)\downarrow$) then $J^X(n) \in W^Y_{f(n)}$ and for all n, $W^Y_{f(n)}$ is finite of cardinality $\leq g(n)$.*

The relation $\leq_{JT}$ is transitive and indeed a weak reducibility [N, 8.4.14]. Further information on weak reducibilities, and jump traceability, may be found in the recent book by Nies [N], especially in Sections 5.6 and 8.6, and 8.4, respectively.

Definition 1.2. *A is* JT-hard *if $\emptyset'$ is jump traceable by A.*
Let $\mathsf{Shigh} = \{Y : Y' \geq_{tt} \emptyset''\}$ *be the class of superhigh sets.*

Theorem 1.3. *Consider the following five properties of a set A.*

(1) A is Turing complete;
(2) A is almost everywhere dominating;
(3) A is JT-hard;
(4) A is superhigh;
(5) A is high.

We have (1)⇒(2)⇒(3)⇒(4)⇒(5), all implications being strict.

Proof. Implications: (1)⇒(2): Dobrinen and Simpson [DS]. (2)⇒(3): Simpson [S] Lemma 8.4. (3)⇒(4): Simpson [S] Lemma 8.6. (4)⇒(5): Trivial, since each truth-table reduction is a Turing reduction.

Non-implications: (2)⇏(1) was proved by Cholak, Greenberg, and Miller [CGM]. (3)⇏(2): By Cole and Simpson [CS], (3) coincides with (4) on the Δ^0_2 sets. But there is a superhigh degree that does not satisfy (2): one can use Jockusch-Shore Jump Inversion for a super-low but not K-trivial set, which exists by the closure of the K-trivials under join and the existence of a pair of super-low degrees joining to $\emptyset'$. (4)⇏(3): We prove in Theorem 2.1 below that there is a jump traceable superhigh degree. By transitivity of $\leq_{JT}$ and the observation that $\emptyset' \not\leq_{JT} \emptyset$, no jump traceable degree is JT-hard. (5)⇏(4): Binns, Kjos-Hanssen, Lerman, and Solomon [BKHLS] proved this using a syntactic analysis combined with a result of Schwartz [S]. □

Historically, the easiest separation (1)(5) is a corollary of Friedberg's Jump Inversion Theorem [F] from 1957. The separation (1)(4) follows similarly from Mohrherr's Jump Inversion Theorem for the tt-degrees [M] (1984), and the separation (4)(5) is essentially due to Schwartz [S] (1982). The classes (2) and (3) were introduced more recently, by Dobrinen and Simpson [DS] (2004) and Simpson [S] (2007).

Notion (3), JT-hardness, may not appear to be very natural. However, Cole and Simpson [CS] gave an embedding of the hyperarithmetic hierarchy $\{0^{(\alpha)}\}_{\alpha<\omega_1^{CK}}$ into the lattice of Π^0_1 classes under Muchnik reducibility making use of the notion of *bounded limit recursive* (BLR) functions. We will see that JT-hardness coincides with BLR-hardness.

Notation. We write

$$\forall n\, f(n) = \lim\nolimits_s^{\text{comp}} \widetilde{f}(n,s)$$

if for all n, $f(n) = \lim_s \widetilde{f}(n,s)$, and moreover there is a computable function $g : \omega \to \omega$ such that for all n, $\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}$ has cardinality less than $g(n)$.

2. Superhighness and jump traceability

In this section we show that superhighness is compatible with the lowness property of being jump traceable, and deduce an answer to a question of Cole and Simpson.

Theorem 2.1. *There is a superhigh jump-traceable set.*

Proof. Mohrherr [M] proves a jump inversion theorem in the tt-degrees: For each set A, if $\emptyset' \leq_{tt} A$, then there exists a set B such that $B' \equiv_{tt} A$. To produce B, Mohrherr uses the same construction as in the proof of Friedberg's Jump Inversion Theorem for the Turing degrees. Namely, B is constructed by finite extensions $B[s] \preceq B[s+1] \preceq \cdots$ Here $B[s]$ is a finite binary string and $\sigma \preceq \tau$ denotes that σ is an initial substring of τ. At stages of the form $s = 2e$ (even stages), one searches for an extension $B[s+1]$ of $B[s]$ such that $J^{B[s+1]}(e)\downarrow$. If none is found one lets $B[s+1] = B[s]$. At stages of the form $s = 2e+1$ (odd stages) one appends the bit $A(e)$, i.e. one lets $B[s+1] = B[s]^\frown\langle A(e)\rangle$. Thus two types of oracle questions are asked alternately for varying numbers e:

(1) Does a string $\sigma \succeq B[s]$ exist so that $J^\sigma(e)\downarrow$, i.e. $B \succeq \sigma$ implies $e \in B'$? (If so, let $B[s+1]$ be the first such string that is found.)
(2) Is $A(e) = 1$?

This allows for a jump trace V_e of size at most 4^e. First, V_0 consists of at most one value, namely the first value $J^\sigma(e)$ found for any σ extending the empty string. Next, V_1 consists of the first value for $\Phi_1^\tau(1)$ found for any τ extending $\langle 0\rangle$, $\langle 1\rangle$, $\sigma^\frown\langle 0\rangle$, $\sigma^\frown\langle 1\rangle$, respectively, in the cases: $0 \notin A$, and $0 \notin B'$; $0 \in A$ and $0 \notin B'$; $0 \notin A$ and $0 \in B'$; and $0 \in A$ and $0 \in B'$. Generally, for each e there are four possibilities: either e is in A or not, and either the extension σ of $B[s]$ is found or not. V_e consists of all the possible values of $J^B(e)$ depending on the answers to these questions.

Hence B is jump traceable, no matter what oracle A is used. Thus, letting $A = \emptyset''$ results in a superhigh jump-traceable set B. □

Question 2.2. *Is there a superhigh set of minimal Turing degree?*

This question is sharp in terms of the notions (1)–(5) of Theorem 1.3: minimal Turing degrees can be high (Cooper [C]) but not JT-hard (Barmpalias [B]).

Cole and Simpson [CS] introduced the following notion. Let A be a Turing oracle. A function $f\colon\ \omega \to \omega$ is *boundedly limit computable by A* if there exist an A-computable function $\widetilde{f} : \omega \times \omega \to \omega$ such that $\lim_s^{\text{comp}} \widetilde{f}(n,s) = f(n)$.

We write

$$\mathsf{BLR}(A) = \{f \in \omega^\omega \mid f \text{ is boundedly limit computable by } A\}.$$

We say that $X \leq_{BLR} Y$ if $\mathsf{BLR}(X) \subseteq \mathsf{BLR}(Y)$. In particular, A is BLR-*hard* if $\mathsf{BLR}(\emptyset') \subseteq \mathsf{BLR}(A)$.

It is easy to see that $\leq_{BLR}$ implies $\leq_{JT}$ (Lemma 6.8 of Cole and Simpson [CS]). The following partial converse is implicit in some recent papers as pointed out to the authors by Simpson.

Theorem 2.3. *Suppose that $A \leq_{JT} B$ where A is a c.e. set and B is any set. Then* $\mathsf{BLR}(A) \subseteq \mathsf{BLR}(B)$.

Proof. Since $A \leq_{JT} B$, by Remark 8.7 of Simpson [S], the function h given by

$$h(e) = J^A(e) + 1 \text{ if } J^A(e) \downarrow,\ h(e) = 0 \text{ otherwise},$$

is B'-computable, with computably bounded use of B' and unbounded use of B. This implies that h is $\mathsf{BLR}(B)$. Let ψ^A be any function partial computable in A. Let g be defined by

$$g(n) = \psi^A(n) + 1 \text{ if } \psi^A(n) \downarrow,\ g(n) = 0 \text{ otherwise}.$$

Letting f be a computable function with $\psi^A(n) \simeq J(f(n))$ for all n, we can use the B-computable approximation to h with a computably bounded number of changes to get such an approximation to g. So g is $\mathsf{BLR}(B)$. By Lemma 2.5 of Cole and Simpson [CS], it follows that $\mathsf{BLR}(A) \subseteq \mathsf{BLR}(B)$. □

Corollary 2.4. *For c.e. sets A, B we have $A \leq_{JT} B \leftrightarrow A \leq_{BLR} B$.*

Corollary 2.5. *JT-hardness coincides with* BLR*-hardness: for all B, $\emptyset' \leq_{JT} B \leftrightarrow \emptyset' \leq_{BLR} B$.*

By Corollary 2.5 and Theorem 1.3((3)⇒(4)), BLR-hardness implies superhighness. Cole and Simpson asked [CS, Remark 6.21] whether conversely superhighness implies BLR-hardness. Our negative answer is immediate from Corollary 2.5 and Theorem 1.3((4)$\not\Rightarrow$(3)).

3. Superhighness, randomness, and K-triviality

We study the class $\mathsf{Shigh}^\diamond$ of c.e. sets that are Turing below all ML-random superhigh sets. First we show that this class contains a promptly simple set.

For background on diagonally non-computable functions and sets of PA degree see [N, Ch 4]. Let λ denote the usual fair-coin Lebesgue measure on $2^{\mathbb{N}}$; a null class is a set $\mathcal{S} \subseteq 2^{\mathbb{N}}$ with $\lambda(S) = 0$.

Fact 3.1 (Jockusch and Soare [JS])**.** *The sets of PA degree form a null class.*

Proof. Otherwise by the zero-one law the class is conull. So by the Lebesgue Density Theorem there is a Turing functional Φ such that $\Phi^X(w) \in \{0, 1\}$ if defined, and

$$\{Z \colon \Phi^Z \text{ is total and diagonally non-computable } \}$$

has measure at least $3/4$.

Let the partial computable function f be defined by: $f(n)$ is the value $i \in \{0, 1\}$ such that for the smallest possible stage s, we observe by stage s that $\Phi^Z(n) = i$ for a set of Zs of measure strictly more than $1/4$. For each n, such an i and stage s must exist. Indeed, if for some n and both $i \in \{0, 1\}$ there is no such s, then $\Phi^Z(n)$ is defined for a set of Zs of measure at most $\frac{1}{4} + \frac{1}{4} = \frac{1}{2} \not\geq \frac{3}{4}$, which is a contradiction. Moreover, we cannot have $f(n) = J(n)$ for any n, because this would imply that there is a set of Zs of measure strictly more than $1/4$ for which Φ^Z is not a total d.n.c. function. Thus f is a computable d.n.c. function, which is a contradiction.

□

Theorem 3.2 (Simpson). *The class* Shigh *of superhigh sets is contained in a* Σ^0_3 *null class.*

Proof. A function f is called diagonally non-computable (d.n.c.) relative to $\emptyset'$ if $\forall x \neg f(x) = J^{\emptyset'}(x)$. Let P be the $\Pi^0_1(\emptyset')$ class of $\{0,1\}$-valued functions that are d.n.c. relative to $\emptyset'$. By Fact 3.1 relative to $\emptyset'$, the class $\{Z\colon \exists f \le_T Z \oplus \emptyset' [f \in P]\}$ is null. Then, since GL_1 is conull, the class

$$\mathcal{K} = \{Z\colon \exists f \le_{\mathrm{tt}} Z' [f \in P]\}$$

is also null. This class clearly contains Shigh. To show that $\mathcal{K}$ is Σ^0_3, fix a Π^0_2 relation $R \subseteq \mathbb{N}^3$ such that a string σ is extended by a member of P iff $\forall u \, \exists v \, R(\sigma, u, v)$. Let $(\Psi_e)_{e\in\mathbb{N}}$ be an effective listing of truth-table reduction procedures. It suffices to show that $\{Z : \Psi_e(Z') \in P\}$ is a Π^0_2 class. To this end, note that

$$\Psi_e(Z') \in P \leftrightarrow \forall x \, \forall t \, \forall u \, \exists s > t \exists v \, R(\Psi_e^{Z'} \upharpoonright_x [s], u, v). \qquad \square$$

Corollary 3.3. *There is no superhigh weakly 2-random set.*

Proof. Let R be a weakly 2-random set. By definition, R belongs to no Π^0_2 null class. Since a Σ^0_3 class is a union of Π^0_2 classes of no greater measure, R belongs to no Σ^0_3 null class. By Theorem 3.2, R is not superhigh. $\square$

To put Corollary 3.3 into context, recall that the 2-random set $\Omega^{\emptyset'}$ is high, whereas no weakly 3-random set is high (see [N, 8.5.21]).

Corollary 3.4. *There is a promptly simple set Turing below all superhigh ML-random sets.*

Proof. By a result of Hirschfeldt and Miller (see [N, Thm. 5.3.15]), for each null Σ^0_3 class $\mathcal{S}$ there is a promptly simple set Turing below all ML-random sets in $\mathcal{S}$. Apply this to the class $\mathcal{K}$ from the proof of Theorem 3.2. $\square$

Next we show that $\mathsf{Shigh}^\Diamond$ is a proper subclass of the c.e. K-trivial sets. Since some superhigh ML-random set is not above $\emptyset'$, each set in $\mathsf{Shigh}^\Diamond$ is a base for ML-randomness, and therefore K-trivial (for details of this argument, see [N, Section 5.1]). It remains to show strictness. In fact in place of the superhigh sets we can consider the possibly smaller class of sets Z such that $G \le_{\mathrm{tt}} Z'$, for some fixed set $G \ge_{tt} \emptyset''$. Let $\mathsf{MLR} = \{R : R \text{ is ML-random}\}$.

Theorem 3.5. *Let* S *be a* Π^0_1 *class such that* $\emptyset \subset S \subseteq \mathsf{MLR}$. *Then there is a* K*-trivial c.e. set* B *such that*

$$\forall G \, \exists Z \in S \, [B \not\le_T Z \ \& \ G \le_{\mathrm{tt}} Z'].$$

Corollary 3.6. *There is a* K*-trivial c.e. set* B *and a superhigh ML-random set* Z *such that* $B \not\le_T Z$. *Thus the class of c.e. sets Turing below all ML-random superhigh sets is a proper subclass of the c.e.* K*-trivials.*

Proof of Theorem 3.5. We assume fixed an indexing of all the Π^0_1 classes. Given an index for a Π^0_1 class P we have an effective approximation $P = \bigcap_t P_t$ where P_t is a clopen set ([N, Section 1.8]).

To achieve $G \le_{\mathrm{tt}} Z'$ we use a variant of Kučera coding. Given (an index of) a Π^0_1 class P such that $\emptyset \subset P \subseteq \mathsf{MLR}$, we can effectively determine $k \in \mathbb{N}$

such that $2^{-k} < \lambda P$. In fact $k \leq K(i) + O(1) \leq 2 \log i + O(1)$ where i is the index for P (see [N, 3.3.3]). At stage t let

$$y_{0,t}, y_{1,t}, \tag{1}$$

respectively be the leftmost and rightmost strings y of length k such that $[y] \cap P_t \neq \emptyset$. Then y_0 is left of y_1 where $y_a = \lim_t y_{a,t}$. Note that the number of changes in these approximations is bounded by 2^k.

Recall that $(\Phi_e)_{e \in \mathbb{N}}$ is an effective listing of the Turing functionals. The following will be used in a "dynamic forcing" construction to ensure that $B \neq \Phi_e^Z$, and to make B K-trivial. Let $c_{\mathcal{K}}$ be the standard cost function for building a K-trivial set, as defined in [N, 5.3.2]. Thus $c_{\mathcal{K}}(x, s) = \sum_{x < w \leq s} 2^{-K_s(w)}$.

Lemma 3.7. *Let Q be a Π^0_1 class such that $\emptyset \subset Q \subset \mathsf{MLR}$. Let $e, m \geq 0$. Then there is a nonempty Π^0_1 class $P \subset Q$ and $x \in \mathbb{N}$ such that either*

(a) *$\forall Z \in P \, \neg \Phi_e^Z(x) = 0$, or*
(b) *$\exists s \, c_{\mathcal{K}}(x, s) \leq 2^{-m}$ & $\forall Z \in P_s^s \, \Phi_{e,s}^Z(x) = 0$,*

where $(P^t)_{t \in \mathbb{N}}$ is an effective sequence of (indices for) Π^0_1 classes such that $P = \lim_t^{\mathrm{comp}} P^t$ with at most 2^{m+1} changes.

The plan is to put x into B in case (b). The change in the approximations P^t is due to changing the candidate x when its cost becomes too large.

To prove the lemma, we give a procedure constructing the required objects.

Procedure $C(Q, e, m)$. Stage s.

(a) Choose $x \in \mathbb{N}^{[e]}$, $x \geq s$.
(b) If $c_{\mathcal{K}}(x, s) \geq 2^{-m}$, GOTO (a).
(c) If $\{Z \in Q_s \colon \neg \Phi_{e,s}^Z(x) = 0\} \neq \emptyset$ let $P^s = \{Z \in Q \colon \neg \Phi_e^Z(x) = 0\}$ and GOTO (b). (In this case we keep x out of B and win.) Otherwise let $P^s = Q$ and GOTO (d). (We will put x into B and win.)
(d) END.

Clearly we choose a new x at most 2^m times, so the number of changes of P^t is bounded by 2^{m+1}.

To prove the theorem, we build at each stage t a tree of Π^0_1 classes $P^{\alpha,t}$, where $\alpha \in 2^{<\omega}$. The number of changes of $P^{\alpha,t}$ is bounded computably in α.

Stage t. Let $P^{\varnothing,t} = S$.

(i) If $P = P^{\alpha,t}$ has been defined let, for $b \in \{0, 1\}$,
$$Q^{\alpha b,t} = P^{\alpha,t} \cap [y_{b,t}],$$
where the strings $y_{b,t}$ are as in (1).
(ii) If $Q = Q^{\beta,t}$ is newly defined let $e = |\beta|$, let m equal n_β (the code number for β) plus the number of times the index for Q^β has changed so far. From now on define $P^{\beta,t}$ by the procedure $C(Q, e, m)$ in Lemma 3.7. If it reaches (d), put x into B.

Claim 1. *(i) For each α the index $P^{\alpha,t}$ reaches a limit P^α. The number of changes is computably bounded in α.*
(ii) For each β the index $Q^{\beta,t}$ reaches a limit Q^β. The number of changes is computably bounded in β.

The claim is verified by induction, in the form $P^\alpha \to Q^{\alpha b} \to P^{\alpha b}$. This yields a computable definition of the bound on the number of changes.

Clearly (i) holds when $\alpha = \emptyset$.

Case $Q^{\alpha b}$: we can compute by inductive hypothesis an upper bound on the index for P^α, and hence an upper bound k_0 on k such that $2^{-k} < \lambda P^\alpha$. If N bounds the number of changes for P^α then $Q^{\alpha b}$ changes at most $N2^{k_0}$ times.

Case P^β, $\beta \neq \emptyset$: Let M be the bound on the number of changes for Q^β. Then we always have $m \le M + n_\beta$ in (ii), so the number of changes for P^β is at most $M2^{M+n_\beta+1}$.

Claim 2. *(i) Let $e = |\beta| > 0$. Then $B \neq \Phi_e(Z)$ for each $Z \in P^\beta$.*

This is clear, since eventually the procedure in Lemma 3.7 has a stable x to diagonalize with.

Given G define $Z \le_T \emptyset' \oplus G$ as follows. For $e > 0$ let $\beta = G\upharpoonright_e$. Use $\emptyset'$ to find the final P^β, and to determine $y_{\beta,b,t}$ ($b \in \{0,1\}$) for $P = P^\beta$ as the strings in (1). Let $y_{\beta,b} = \lim y_{\beta,b,t}$.

Note that $y_\gamma \prec y_\delta$ whenever $\gamma \prec \delta$. Define Z so that $y_{G(e)} \prec Z$.

For $G \le_{\mathrm{tt}} Z'$ define a function $f \le_T Z$ such that $G(e) = \lim_s^{\mathrm{comp}} f(e,s)$ (i.e., a computable bounded number of changes). Given e, to define $f\upharpoonright_e [s]$ search for $t > s$ such that $y_{\alpha,t} \prec Z$ for some α of length e, and output α.

$\square$

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