CODING IN THE PARTIAL ORDER OF ENUMERABLE SETS

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Abstract. We develop methods for coding with first-order formulas into the partial order $\mathcal{E}$ of enumerable sets under inclusion. First we use them to reprove and generalize the (unpublished) result of the first author that the elementary theory of $\mathcal{E}$ has the same computational complexity as the theory of the natural numbers. Relativized versions of the coding methods show that the p.o. of Σ_p^0 and Σ_q^0 sets are not elementarily equivalent for natural numbers $p \neq q$. As a further application, definability of the class of quasimaximal sets in $\mathcal{E}$ is obtained. On the other side, we prove theorems limiting coding and definability in $\mathcal{E}$, thereby establishing a sharp contrast between $\mathcal{E}$ and other structures occurring in computability theory.

1. Introduction.

The notion of recursively enumerable (in the following called *enumerable*) set is fundamental for logic and mathematics. For example, enumerable sets arise as word problems of finitely generated subgroups of finitely presented groups and as solution sets for Diophantine equations, as well as in the study of elementary theories. In the following, we restrict ourselves to enumerable sets of natural numbers (while any domain of countably many effectively given objects, like formulas in a first-order language or reduced words in a finitely generated free group, would be acceptable as well). Relating enumerable sets in the most elementary way, namely via the inclusion relation, one obtains the partial order $\mathcal{E}$. Despite of the conceptually simple way $\mathcal{E}$ was introduced, it is a distributive lattice of great algebraic complexity. Several interrelated directions in the study of $\mathcal{E}$ have been followed: one is the investigation of automorphisms (initiated in [So 74]), a further one is the relationship between the behavior of an enumerable set as an element of $\mathcal{E}$ and its computational complexity (see e.g. [Ma 66] and [Ha,So 91]). Here we follow another approach, the approach of studying definability and coding.

Definability and coding are principal concerns in the study of all structures arising from computability theory, for instance also for degree structures like the p.o. of r.e. Turing degrees. In an analysis of a structure by coding methods, typically, first the uniform coding in a structure $\mathcal{A}$ of a sufficiently complex class of structures

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(say the class of finite partial orders) is investigated to obtain undecidability of the elementary theory of $\mathcal{A}$. After that, coding of a standard model of arithmetic is used to determine the complexity of the theory. In many cases, the result was obtained that the theory has the highest possible complexity, namely the same as true arithmetic. For instance, in [N 94] the second author proves the result for the structure of enumerable many–one degrees. To do so, a method is introduced to obtain definability with parameters in the given structure of sets which have a Σ^0_k index set, by using induction over k. The first author combined the method with $\mathcal{E}$-specific techniques to obtain the "Ideal definability lemma" (see below), which is of central importance for the coding results obtained here.

In computability theory, first-order definability without parameters is studied especially to investigate the relationship between external concepts (like a reducibility in the context of the p.o. of enumerable sets) and subclasses of the structure which can be expressed from within the structure, without reference to external concepts. A definability result shows that the external concept is in fact unnecessary to determine the property of enumerable sets in question. Examples of such results are the definability of the class of m-complete sets in $\mathcal{E}$, proved by L. Harrington (see [So 87]) and a recent result of Nies, Shore and Slaman which implies that that the class Low_2 and many similar classes are definable in the p.o. of enumerable Turing-degrees ([N,S,Sl ta]).

We now describe the coding methods used to some detail. Uniform coding of a class $\mathbf{C}$ of structures for a finite relational language in a structure $\mathcal{A}$ relies on a scheme (decoding key) of formulas with parameters: a formula $\varphi_U(x;\overline{p})$ to define the universe of a structure in $\mathbf{C}$, and formulas $\varphi_R(x_1,\dots,x_n;\overline{p})$ for each n-ary relation symbol R of L (including equality) such that for each structure in $\mathbf{C}$, an isomorphic copy is defined on $\{x : \mathcal{A} \models \varphi_U(x;\overline{c})\}$ within $\mathcal{A}$ for an appropriate parameter list $\overline{c}$. The standard indirect method to prove undecidability of $\mathrm{Th}(\mathcal{A})$ proceeds by uniformly coding a class $\mathbf{C}$ such that $\mathrm{Th}(\mathbf{C})$ is hereditarily undecidable. For degree structures, both the class $\mathbf{C}$ used and the coding scheme was fairly simple (in terms of the complexity of the formulas used). For instance, to show that the theory of the structure of r.e. m-degrees is undecidable one can use the class of finite distributive lattices (viewed as p.o.): each such lattice is isomorphic to an initial interval $[\mathbf{0},\mathbf{a}]$ of the r.e. m-degrees. However, in both known proofs of undecidability for $\mathrm{Th}(\mathcal{E})$, $\mathbf{C}$ was the same complicated class containing infinite structures, namely the class of recursive Boolean pairs. (Since a standard model of arithmetic can be coded in an appropriate recursive Boolean pair by extending methods in [Bu,McK 81], this already gives a rather indirect way to code a standard model of arithmetic in $\mathcal{E}$.) Here we introduce a more direct coding of a standard model of arithmetic. We make use of a main technical result due to the first author. For an r.e. set A, let $\mathcal{B}(A)$ be the Boolean algebra of components of r.e. splittings of A, and let $\mathcal{R}(A)$ be the ideal of $\mathcal{B}(A)$ consisting of the recursive subsets of A. An ideal I of $\mathcal{B}(A)$ is called *k-acceptable* if $\mathcal{R}(A) \subseteq I$ and $\{e : W_e \in I\}$ is Σ^0_k. Harrington's Ideal Definability Lemma states that, for given odd $k \geq 3$, each k-acceptable ideal of $\mathcal{B}(A)$ can be defined with parameters in a uniform way. As in [N94], the result is proved by induction, here over odd $k \geq 3$. Our direct coding of a standard model of arithmetic allows a substantial simplification of the first author's proof that true arithmetic can be interpreted in $\mathrm{Th}(\mathcal{E})$. As in many proofs of the similar result for other structures, one obtains a class of uniformly coded *standard* models of arithmetic which can be recognized by a first-order condition on their codes, the parameters.

Beyond determining the complexity of the elementary theory of $\mathcal{E}$, the methods can be used for definability results and for obtaining elementary differences between relativized versions of $\mathcal{E}$. E. Herrmann (personal communication) asked if, for $0 < p < q$, the relativization of $\mathcal{E}$ to $\emptyset^{(p-1)}$ (i.e. the Σ^0_p-sets) and to $\emptyset^{(q-1)}$ are elementarily equivalent. Evidence for an affirmative answer came from the fact that constructions of r.e. sets which show that $\mathcal{E}$ possesses certain first-order properties (like the construction of a maximal set in [Fr 58]) relativize and therefore show that $\mathcal{E}^Z$ has the same property for each $Z \subseteq \omega$.

However, we answer Herrmann's question negatively. An elementary difference between the p.o. of Σ^0_p- and Σ^0_q-sets $(0 < p < q)$ is obtained by considering the "coding power" of a certain scheme of formula in the structure, which increases with the complexity of the oracle $\mathcal{E}$ is relativized to. We use two facts:

a) the proof that there is an interpretation of true arithmetic relativizes to $\mathcal{E}^Z$; in particular, also in $\mathcal{E}^Z$ there is a first-order recognizable class of coded standard models of arithmetic.
b) the proof of the Ideal Definability Lemma is strict, namely for odd $k \geq 3$ one obtains a formula defining precisely the k-acceptable ideals of $\mathcal{B}(A)$ as the parameters vary. With the obvious relativization of the notion of k-acceptability to $Z \subseteq \omega$ (requiring that $\{e : W^Z_e \in I\}$ is $\Sigma^0_k(Z)$) the similar result holds for $\mathcal{E}^Z$.

Fix a Σ^0_{q+c}-set S which is not Σ^0_{p+c}, for some sufficiently large c. We obtain the elementary difference by expressing that some formula obtained from the Ideal Definability Lemma codes S, viewed as a subset of a member of our class of coded standard models of arithmetic. This holds in the Σ^0_q-sets but not in the Σ^0_p-sets since, by strictness of the Ideal Definability Lemma, in this case S would be Σ^0_{p+c}. In [N 95], the second author develops the method applied above for separating relativizations in more generality. In that paper, the result is re-obtained as an application of the "Separation Theorem". It can also be read as a survey–form introduction into coding in various structures arising from recursion theory, including $\mathcal{E}$. Moreover, in [N ta2], the second author proves an Ideal Definability Lemma for certain ideal lattices of enumerable Boolean algebras, and uses this to prove that the theory of arbitrary intervals of $\mathcal{E}$ which are not Boolean algebras interprets true arithmetic.

Recall that $\mathcal{L}^*(A)$ is the lattice of r.e. supersets of A modulo finite differences and that A is called *quasimaximal* if $\mathcal{L}^*(A)$ is finite or, equivalently, if A is the intersection of finitely many maximal sets. In [So 87] it is asked if the class of quasimaximal sets is definable in $\mathcal{E}$. We answer this question affirmatively. One way is to analyze our coding of standard models of arithmetic. Alternatively, the definability of quasimaximality and further classes of hh-simple sets can be obtained from the Ideal Definability Lemma and certain isomorphism properties of Boolean algebras which are coded in $\mathcal{E}$ with parameters (Theorem 4.3).

In the last section we investigate the limits of definability and coding. We show that no infinite linear order can be coded (without parameters) even in the most general way, namely on equivalence classes of n-tuples. An example of a coding of that kind is the coding of $\mathbb{Q}$ in $\mathbb{Z}$, where a rational is represented by an equivalence class of ordered pairs (fractions). The proof makes use of the fact that for each partition of ω into three infinite recursive sets R, S, T there is a canonical isomorphism

$\mathcal{E} \to \mathcal{E}^3$ given by $X \to (X \cap R, X \cap S, X \cap T)$, combined with a model theoretic result due to Feferman and Vaught [FV 59] that a first-order property of a tuple in a model of the form $\mathbf{A}^n$ can be expressed as a certain Boolean combination of first-order properties of the components. First we prove a noncoding theorem in the context of uniform first–order definability with parameters, which can be considered as a weak form of the model–theoretic notion of stability for $\mathcal{E}$: there is no uniform way to define, even with parameters, a linear order on arbitrarily large classes $\{R_1, \ldots, R_k\}$ of pairwise disjoint recursive sets. This implies that no linear order can be defined in a first-order way on the atoms of $\mathcal{L}^*(A)$, if $\mathcal{L}^*(A)$ is a Boolean algebra with infinitely many atoms.

Hodges and the second author ([Ho,N ta]) have recently shown that the non-codability of infinite linear orders holds in fact for any structure of the form $\mathbf{A} \times \mathbf{A}$. However, the proof given here contains interesting insights into further self-similarity properties of $\mathcal{E}$ and also gives an effective upper bound on the cardinality of a l.o. which can be coded by a given formula.

Recall that $\mathcal{E}^*$ is the p.o. of r.e. sets modulo finite differences. Both $\mathcal{E}$ and $\mathcal{E}^*$ are distributive lattices (however, for definability and coding concerns, it does not matter which language is used, unless one is interested in low-level fragments of the theory). We state our results for $\mathcal{E}$ instead of $\mathcal{E}^*$ mostly for notational convenience: from the methods in [La 68] one can derive that, if $C \subseteq \mathcal{E}^n$ is closed under finite variants, then

$$C \text{ definable in } \mathcal{E} \Leftrightarrow C/=^* \text{ definable in } \mathcal{E}^*,$$

and similarly for definability with parameters. Now our coding and definability results do not refer to membership of particular elements. So one can easily transfer all the results to $\mathcal{E}^*$, e.g. to prove that $\{A^* : \mathcal{L}^*(A) \text{ finite}\}$ is definable in $\mathcal{E}^*$ or that the Σ^0_2-sets modulo finite variants are not elementarily equivalent to $\mathcal{E}^*$.

Note that a structure $\mathbf{A}$ can be coded in $(\omega, +, \times)$ iff there is an onto map $e : \omega \to A$ such that the preimages of the relations and functions of A are arithmetical. For instance, if $\mathbf{A}$ is $\mathcal{E}^*$, let $e(i) = W_i^*$. Each relation on $\mathbf{A}$ which is definable must be invariant under automorphisms of $\mathbf{A}$ and have an arithmetical preimage under e. The questions arises if a "maximum definability property" holds, namely if these two properties actually characterize the definable relations. The question has been answered affirmatively for the structure of Δ^0_2 T-degrees [Sl,W ta]. For $\mathcal{E}^*$, S. Lempp [Lem 87] shows that some natural property of elements of $\mathcal{E}^*$ is invariant but not definable, but this property does not correspond to an arithmetical index set. Selivanov [Se 89] gives a counterexample for $\mathcal{E}$ which however is not invariant under finite differences.

Let $n(A) = k$ if $\mathcal{L}^*(A)$ is a Boolean algebra with k atoms and $n(A) = -1$ else. From the noncoding result for linear orders it follows that $\{\langle A^*, B^* \rangle : n(A) < n(B)\}$ is a counterexample to the maximum definability property. In ([N ta1]) the second author gives a counterexample which is a subclass of $\mathcal{E}^*$. As further nondefinability results, we show that "X^* automorphic to Y^*" and "$\mathcal{L}^*(X) \cong \mathcal{L}^*(Y)$" cannot be defined in $\mathcal{E}^*$. There are natural recursion theoretic structures where "x automorphic to y" can be defined , for instance by a result of Slaman and Woodin, the p.o. of Δ^0_2 Turing degrees.

We now review the notation used: capital letters A, B, C, X, Y range over r.e. sets, letters R, S, T over recursive sets. Let $X \sqsubset A \Leftrightarrow (\exists Y)[X \cap Y = \emptyset \wedge X \cup Y = A]$, $\mathcal{B}(A) = \{X : X \sqsubset A\}$ and $\mathcal{R}(A) = \{R : R \sqsubset A\}$. An ideal I of $\mathcal{B}(A)$ is k-acceptable if $\mathcal{R}(A) \subseteq I$ and $\{e : W_e \in I\}$ is Σ^0_k. If we say "I is acceptable" we mean that I is k-acceptable, where k is a fixed number which depends only on the context in which I is defined (e.g. on formulas in some coding scheme or on arithmetical constructions).

Given an r.e. set A define a Δ^0_3-enumeration $(U_e)_{e\in\omega}$ of $\mathcal{B}(A)$ as follows: if $e = \langle i, j\rangle$, $W_i \cap W_j = \emptyset$ and $W_i \cup W_j = A$ let $U_e = W_i$ and write $\overline{U}_e$ for W_j. Else let $U_e = \emptyset$ and $\overline{U}_e = A$.

An interpretation of a theory T_1 in a theory T_2 is a many–one reduction of T_1 to T_2 via a map which is defined on sentences in the language of T_1 in some natural way.

2. Tools for Coding in $\mathcal{E}$.

2.1 Ideal Definability Lemma [Harrington].
For each $n \geq 1$ there is a formula with parameters $\varphi_n(X; A, \overline{C})$ ($|\overline{C}| = n$) such that, if A is non-recursive, for varying $\overline{C}$, $\{X : \mathcal{E} \models \varphi_n(X; A, \overline{C})\}$ ranges precisely over the class of ideals I of $\mathcal{B}(A)$ which contain $\mathcal{R}(A)$ and have a Σ^0_{2n+1} index set (i.e. the $2n+1$-acceptable ideals).

Proof. The formulas φ_n are defined by induction over n, thereby reducing the problem of defining an $2n+3$-acceptable ideal to the problem to define an $2n+1$-acceptable one. Here we verify that each class $\{X : \mathcal{E} \models \varphi_n(X; A, \overline{C})\}$ is an ideal of the described sort. In the appendix, we complete Harrington's proof by showing that, conversely, each ideal of $\mathcal{B}(A)$ of the described kind can be defined with appropriate parameters.

For $n = 1$, let

$$\varphi_1(X; A, C) \equiv X \sqsubset A \wedge (\exists R \subseteq A)[\ X \subseteq C \cup R].$$

Clearly, for each C, an ideal of $\mathcal{B}(A)$ containing $\mathcal{R}(A)$ is defined via φ_1. Moreover, because "$X \sqsubset A$" and "R recursive" is Σ^0_3 on indices of r.e. sets and "$X \subseteq C \cup R$" is Π^0_2, such an ideal must have Σ^0_3 index set.

For the inductive step, if $\overline{C} = (C_0, \dots, C_{n-1})$, let

$$\begin{aligned} &\varphi_{n+1}(X; A, \overline{C}, C_n) \equiv X \sqsubset A \wedge \\ (1) \qquad &(\exists R \subseteq A)(\forall S \subseteq A - R) \\ &\varphi_n(X \cap S \cap C_n; C_n, \overline{C}). \end{aligned}$$

(Recall that the variables R,S range over recursive sets.) Firstly, if $X \sqsubset A$ is recursive, then (1) holds via $R = X$. Secondly, the class of X satisfying φ_n is downward closed, and if X, Y satisfy φ_{n+1} via R_X and R_Y respectively, then $X \cup Y$ satisfies φ_{n+1} via $R_X \cup R_Y$, by inductive hypothesis on φ_n. Finally, to see that φ_{n+1} defines only Σ^0_{2n+3}-ideals, we write (1) more explicitly (for the moment, R, S range over arbitrary sets):

$$\begin{aligned} &(\exists R \subseteq A)(\exists \tilde{R})[R \cap \tilde{R} = \emptyset\ \wedge R \cup \tilde{R} = \omega\ \wedge \\ &(\forall S \subseteq A \cap \tilde{R})[S \text{ nonrecursive } (\Pi_3) \vee \\ &\varphi_n(X \cap S \cap C_n, C_n, \overline{C})(\Sigma^0_{2n+1})]. \end{aligned}$$

Because $n \geq 1$, this shows that the corresponding index set is Σ^0_{2n+3}. □

Harrington's result, as well as this argument, relativize to any oracle Z. Hence, in $\mathcal{E}^Z$, φ_n defines precisely those ideals $I \lhd \mathcal{B}(A)$ containing $\mathcal{R}(A)$ such that $\{i : W_i^Z \in I\}$ is $\Sigma^0_{2n+1}(Z)$.

Our goal for the rest of this Section is to introduce a *coding configuration*, i.e. a framework for coding all arithmetical relational structures in $\mathcal{E}$. The coding configuration will consists of an r.e. set A and an acceptable ideal I of $B(A)$ such that $\mathcal{B}(A)/I$ possesses infinitely many atoms P_k/I $(k \in \omega)$. The atom P_k/I is thought to represent the number k . We show that for any A such that $\mathcal{L}^*(A)$ is a Boolean algebra with infinitely many atoms, and for an appropriate choice of I, the atoms of $\mathcal{L}^*(A)$ can be used to represent pairs of atoms of $\mathcal{B}(A)/I$. Note that an atom of $\mathcal{L}^*(A)$ has the form $M \cup A$ for some recursive M which is unique modulo $\mathcal{R}(A)$. It is possible to code an edge relation on the atoms of $\mathcal{B}(A)/I$ by considering intersections $M \cap P_k$. More precisely, it is shown that for fixed p, each Σ^0_p-relation on $\{P_k/I : k \in \omega\}$ can be uniformly defined with parameters. The relations are encoded by further acceptable ideals of $\mathcal{B}(A)$.

From now on, fix an (otherwise arbitrary) r.e. set A such that $\mathcal{L}^*(A)$ is a Boolean algebra with infinitely many atoms. Let $\mathbf{M}_0$ be the class of recursive sets M such that $(A \cup M)^*$ is an atom of $\mathcal{L}^*(A)$. For $M, N \in \mathbf{M}_0$ write $M \sim N$ if $A \cup M =^* A \cup N$.

We will show that for each such A a coding configuration of the desired kind exists. For better understanding, first we consider a simplified version of the coding configuration, at the cost of obtaining coding of a given arithmetical structure only in the structure in $\mathcal{E}$ with an additional unary predicate. Obtain a u.r.e. partition $(P_k)_{k\in\omega}$ of A by modifying the proof of the Friedberg Splitting Theorem in [So 87] so that a splitting of A into infinitely many sets is produced. In this simplified version, the number $k \in \omega$ is represented by P_k. By the argument in [So 87], for each r.e. W and each k,

$$W - A \text{ non r.e.} \Rightarrow W - P_k \text{ non r.e.}$$

In particular, $M - P_k$ is non-r.e. for each $M \in \mathbf{M}_0$, and hence $M \cap P_k$ is not recursive. In our approximation, we use a unary predicate symbol for the subclass $\{P_k : k \in \omega\}$ in the coding. This subclass coincides with the universe of the structure to be coded.

In an arithmetical way, for each pair P_i, P_j fix $M_{i,j} \in \mathbf{M}_0$ (representing this pair) in a way that different pairs are represented by sets in $\mathbf{M}_0$ which are different modulo $\sim$. Given an arithmetical binary relation E, we define a copy of E on $\{P_k : k \in \omega\}$ using two acceptable ideals J_0, J_1. Let J_0 $[J_1]$ be the ideal generated by $\mathcal{R}(A)$, all sets $M \cap P_k$ such that M is not equivalent to some $M_{i,j}$ modulo $\sim$, and the sets $M_{i,j} \cap P_k$ such that either $\neg Eij$, or Eij but $k \neq i$ $[k \neq j]$. Then one can recover E from J_0, J_1 because

$$Eij \Leftrightarrow (\exists M \in \mathbf{M}_0)[M \cap P_i \notin J_0 \ \wedge \ M \cap P_j \notin J_1]. \tag{2}$$

This can be verified using the facts that $M \cap P_k \notin \mathcal{R}(A)$ for each m, k, and, for $M, N \in \mathbf{M}_0$, either $M \cap P_k$, $N \cap P_k$ are disjoint on the complement of set in $R(A)$ or they are equal.

Now, with an additional unary predicate for $\{P_k : k \in \omega\}$, a copy of E on this set can be defined with parameters by (2), since $\mathbf{M}_0$, J_0 and J_1 are definable with parameters.

As already mentioned, in the full coding configuration, a number k is not represented by P_k but by an equivalence class P_k/I in some quotient algebra $\mathcal{B}(A)/I$, I an acceptable ideal. To ensure that the set of objects representing numbers is a definable set modulo the definable equivalence relation on elements of $\mathcal{B}(A)$ given by I, we construct I in a way that $(P_k/I)_{k\in\omega}$ enumerates the atoms of $\mathcal{B}(A)/I$ (without repetition). By imposing further conditions on I, the ability to code arithmetical relations on the objects representing numbers can be maintained.

To ensure that the atoms of $\mathcal{B}(A)/I$ are precisely the elements P_k/I, we build a sequence of uniformly p-acceptable maximal ideals I_k of $\mathcal{B}(P_k)$ (p some fixed number) and let

$$I = \{U \subset A : (\forall k)[U \cap P_k \in I_k]\}. \tag{3}$$

Note that I is arithmetical and contains the ideal of $\mathcal{B}(A)$ generated by all the ideals I_k.

To encode an arithmetical binary (say) relation E on $\{P_k/I : k \in \omega\}$, we must turn the right hand side in (2) into a coding formula $\psi(P,Q;\overline{D})$ on $\{P : P/I \text{ atom in } \mathcal{B}(A)/I\}$, which only depends on equivalence classes modulo I, so that the corresponding relation on equivalence classes P_k/I is a copy of E. Suppose that, instead of the sets $M_{i,j}$, there is a sequence of sets $M = M_{i,j,X,Y} \in \mathbf{M}_0$ ($i,j \in \omega, X,Y \in I$) which are pairwise distinct modulo $\sim$ so that $M \cap P_i - X$ and $M \cap P_j - Y$ are non-recursive. Define J_0, J_1 as before with $M_{i,j,X,Y}$ instead of $M_{i,j}$. Let the desired coding formula ($\psi(P,Q;\overline{D})$ be obtained by expressing in the language of $\mathcal{E}$ with a list of parameters $\overline{D}$ that

$$(\forall X,Y \in I)(\exists M \in \mathbf{M}_0)[M \cap P - X \notin J_0 \wedge M \cap Q - Y \notin J_1].$$

Then it can be shown (using particular properties of I) that, for any i,j and any P,Q such that $P/I = P_i/I$ and $Q/I = P_j/I$,

$$Eij \Leftrightarrow \mathcal{E} \models \psi(P,Q;\overline{D}).$$

We now formally introduce and prove the existence of coding configurations.

2.2 Lemma. *Suppose A is an r.e. set such that $\mathcal{L}^*(A)$ is a Boolean Algebra with infinitely many atoms. Let $\mathbf{M}_0$ be the class of recursive sets M such that $(A \cup M)^*$ is an atom in $\mathcal{L}^*(A)$. Then there exist*

a) *a p-acceptable (for some fixed p) ideal I of $\mathcal{B}(A)$ and a r.e. sequence $(P_k)_{k\in\omega}$ of pairwise disjoint sets in $\mathcal{B}(A)$ such that all P_k/I are atoms in $\mathcal{B}(A)/I$ and each atom in $\mathcal{B}(A)/I$ is represented by precisely one P_k.*

b) *sets $M_{i,j,X,Y} \in \mathbf{M}_0$ ($i,j \in \omega, X,Y \in I$) which can be obtained recursively in some oracle $\emptyset^{(c)}$ from i,j and indices for X,Y and are pairwise distinct modulo $\sim$ such that*

$$M_{i,j,X,Y} \cap P_i - X \text{ and } M_{i,j,X,Y} \cap P_j - Y$$

are non-recursive.

We call $A, (P_k)_{k\in\omega},\ I, (M_{i,j,X,Y})_{i,j\in\omega,X,Y\in I}$ *a* coding configuration (based on A).

Proof. The sequence (P_k) can be chosen to be any u.r.e. partition of A obtained by proving a version of the Friedberg Splitting Theorem for a splitting into infinitely many sets. We modify the proof in [So 87] of that theorem in the desired way. By the argument in [So87] for each r.e. W and each k, $W - A$ non r.e. $\Rightarrow W - P_k$ non r.e. In particular, $M - P_k$ is not r.e. for each $M \in \mathbf{M}_0$.

The ideal I is determined from a sequence (I_k) of uniformly p-acceptable maximal ideals of $\mathcal{B}(P_k)$ by (3). In general, if P is r.e. non-recursive, an acceptable maximal ideal J of $\mathcal{B}(P)$ can be constructed as follows. Let $(U_e)_{e\in\omega}$ be a Δ^0_3-listing of $\mathcal{B}(P)$ as described at the end of section 1. One builds an ascending sequence $(X_k)_{k\in\omega}$ of elements of $\mathcal{B}(P)$ which generate J, ensuring that

$$(\forall e)[U_e \in J \vee \overline{U}_e \in J]$$

(to make J maximal) and

$$(\forall k)[P - X_k \text{ nonrecursive}]$$

(to ensure $J \neq \mathcal{B}(P)$). The process of defining the (X_k) must be recursive in some $\emptyset^{(c)}$. Let $X_0 = \emptyset$. Inductively, for $k > 0$, one has to make a decision, recursively in $\emptyset^{(c)}$, if

$$X_n = X_{n-1} \cup U_n \text{ or } X_n = X_{n-1} \cup \overline{U}_n. \tag{4}$$

If one of these sets has a recursive complement R in P, one has to take the other (i.e. R is added to X_{n-1}). If both are non-recursive, one can decide either way. In particular, if $\overline{U}_n$ is recursive, then the first set has the recursive complement $\overline{X}_{n-1} \cap \overline{U}_n$, so $\overline{U}_n \in J$. This shows $\mathcal{R}(P) \subseteq J$, so J is acceptable.

To meet (b), simultaneously with the ideals I_k, one builds a descending sequence $(\mathbf{M}_r)_{r\geq 1}$ of uniformly arithmetical subclasses of $\mathbf{M}_0$ such that $\mathbf{M}_r/_\sim$ is infinite for each r and

$$(\forall k)(\forall X \in I)(\exists r)(\forall M \in \mathbf{M}_r)[M \cap P_k - X \text{ nonrecursive}]. \tag{5}$$

The decision (4) is made in a way to ensure that $\mathbf{M}_r/\sim$ is infinite (see below). If (5) holds, then, to define the sequence $M_{i,j,X,Y}$ in (b), we work by induction on codes for quadruples consisting of i, j and (indices for) X, Y. Given such a quadruple, in an arithmetical way determine r such that $\mathbf{M}_r$ satisfies (5) for both i, X and j, Y (this is possible since the classes $\mathbf{M}_p$ form a descending chain). Since $\mathbf{M}_r/\sim$ is infinite, one can determine a set $M_{i,j,X,Y} \in \mathbf{M}_r$ which is distinct under $\sim$ from the previously chosen sets.

Let (U^k_n) be a Δ^0_3 double sequence such that for each k, $(U^k_n)_{n\in\omega}$ is a listing of $\mathcal{B}(P_k)$. We define generating sequences $(X^k_n)_{n\in\omega}$ for I_k and the descending sequence $(\mathbf{M}_r)$ in a way that, if $r = \langle k, n\rangle$, then

$$(\forall M \in \mathbf{M}_{\langle k,n\rangle})[(M - P_k) \cup X^k_n \text{ non r.e.}]. \tag{6}$$

This will suffice to meet (5). Let $X^0_0 = \emptyset$ and $\mathbf{M}_{\langle 0,0\rangle} = \mathbf{M}_0$. Then (6) holds for $\mathbf{M}_{\langle 0,0\rangle}$ by the definition of $\mathbf{M}_0$. In step $r = \langle k, n\rangle > 0$ of the construction, we do the following.

- If $n = 0$, we let $X^k_0 = \emptyset$ and $\mathbf{M}_p = \mathbf{M}_{p-1}$. Then (6) holds for k, n.
- If $n > 0$, we have to decide if $X^k_n = X^k_{n-1} \cup U^k_n$ or $X^k_n = X^k_{n-1} \cup \overline{U}^k_n$.

By (6) for $k, n-1$, for each $M \in \mathbf{M}_{r-1}$ $(\subseteq \mathbf{M}_{\langle k,n-1\rangle})$, $(M-P_k) \cup X^k_{n-1} \cup U^k_n$ or $(M-P_k) \cup (X^k_{n-1} \cup \overline{U}^k_n)$ is non r.e. The question which set is non r.e. only depends on $M/_\sim$. We let $X^k_n = X^k_{n-1} \cup U^k_n$ if the first case applies for infinitely many $M \in \mathbf{M}_{p-1}$ (modulo $\sim$), and $X^k_n = X^k_{n-1} \cup \overline{U}^k_n$ else. Moreover, we let $\mathbf{M}_r$ be the class of sets M in $\mathbf{M}_{r-1}$ such that $(M-P_k) \cup X^k_n$ is non r.e. Note that, if (say) $X^k_{n-1} \cup U^k_n$ has a recursive complement R in P_k, then $(M-P_k) \cup (X^k_{n-1} \cup U^k_n) = M \cap (\omega - R)$ is r.e. for each $M \in \mathbf{M}_0$, so we automatically define $X^k_n = X^k_{n-1} \cup \overline{U}^k_n$, as required in the general procedure described above. Thus I_k is a maximal ideal of $\mathcal{B}(P_k)$ containing $\mathcal{R}(P_k)$. Moreover the index set of I_k is arithmetical, as the decision above can be carried out recursively in some $\emptyset^{(c)}$, $c \in \omega$.

Now let $I = \{U \sqsubset A : (\forall k)(U \cap P_k \in I_k]\}$. We verify (a). Since the ideals I_k are uniformly p-acceptable in $\mathcal{B}(P_k)$ for some fixed p and $\mathcal{R}(A) \subseteq I$, I is an acceptable ideal in $\mathcal{B}(A)$. Moreover $I \cap \mathcal{B}(P_k) = I_k \cap \mathcal{B}(P_k)$ for each k. Since $|\mathcal{B}(P_k)/I_k| = 2$, this implies that P_k/I is an atom in $\mathcal{B}(A)/I$. If $U \in \mathcal{B}(A) - I$, then, for some $k, P_k \cap U \notin I_k$. So $P_k - U \in I$ by the maximality of I_k in $\mathcal{B}(P_k)$, i.e. $P_k/I \leq U/I$. This implies that each atom in $\mathcal{B}(A)/I$ is of the form P_k/I.

Finally we verify that the property (6) we ensured in the construction implies (5). Let $X \in I$ and $k \in \omega$ be arbitrary, and let n be a number such that $X \cap P_k \subseteq X^k_n$. Let $r = \langle k, n\rangle$, and assume for a contradiction that, for $M \in \mathbf{M}_r$, $M \cap P_k - X$ is recursive. Then $R = M \cap P_k - X^k_n$, as an element of $\mathcal{B}(P_k)$ contained in a recursive set, is also recursive. Since $(M - P_k) \cup X^k_n = (M \cap \overline{R}) \cup X^k_n$ this contradicts (6).□

2.3 Coding Lemma. *Fix a coding configuration and let $p \geq 0$ and $n \geq 1$. Then, for each Σ^0_p-relation $E \subseteq \omega^n$, the canonical copy of E on the set $\{P_k/I : k \in \omega\}$ can be defined from parameters in a uniform way.*

Proof. For notational convenience, assume that $n = 2$. As explained above, we need to give a formula with parameters $\psi(P, Q; \overline{D})$ such that, for any binary Σ^0_p relation E a list of parameters $\overline{D}$ exists with the property that, for atoms P/I, Q/I of $\mathcal{B}(A)/I$,

$$\mathcal{E} \models \psi(P, Q; \overline{D}) \Leftrightarrow (\exists i, j)[Eij \wedge P_i/I = P/I \wedge P_j/I = Q/I]. \tag{7}$$

Let J_0 be the ideal of $\mathcal{B}(A)$ generated by $\mathcal{R}(A)$, the classes

$$\{M \cap P_k : M \in \mathbf{M}_0 \wedge (\forall i, j, X, Y)(\neg M \sim M_{i,j,X,Y}]\}$$

and

$$\{M_{i,j,X,Y} \cap P_k : \neg Eij \vee (Eij \wedge k \neq i)\}.$$

Define an ideal J_1 of $\mathcal{B}(A)$ in a similar way, but replacing the third generating class by

$$\{M_{i,j,X,Y} \cap P_k : \neg Eij \vee (Eij \wedge k \neq j)\}.$$

We claim that the first order formula $\psi(P, Q; \overline{D})$ expressing the following satisfies (7):

$$\begin{aligned}&(\forall X, Y \in I)(\exists M)[M \in \mathbf{M}_0 \wedge \\ &\quad M \cap P - X \notin J_0 \wedge M \cap Q - Y \notin J_1].\end{aligned} \tag{8}$$

(Note that the matrix of (8) only depends on M modulo $\sim$.) To show (7), suppose that P/I and Q/I are atoms. Choose i, j such that $P/I = P_i/I$ and $Q/I = P_j/I$. First suppose that Eij holds. We show $\mathcal{E} \models \psi(P, Q; \overline{D})$. Given $X, Y \in I$, let

$$\tilde{X} = X \cup (P_i - P), \ \ \tilde{Y} = Y \cup (P_j - Q),$$

and let

$$M = M_{i,j,\tilde{X},\tilde{Y}}.$$

We claim that $\psi(P, Q; \overline{D})$ holds via M. Assume for a contradiction that, say, $(M \cap P) - X \in J_0$. Since $(M \cap P_i) - \tilde{X} = M \cap (P_i \cap P) - X \subseteq (M \cap P) - X$, $(M \cap P_i) - \tilde{X} \in J_0$. Because Eij holds, this means that $(M \cap P_i) - \tilde{X}$ is contained in the union of a recursive subset of A and a finite union of sets of the form $M' \cap P_{i'}$, $M' \in \mathbf{M}_0$, $i' \neq i$ or $M' \not\sim M$. Because the (P_k) are pairwise disjoint and $M' \cap M \in \mathcal{R}(A)$ for $M' \not\sim M$, $(M \cap P_i) - \tilde{X}$ is contained in a recursive subset of A and hence, as an element of $\mathcal{B}(A)$, is recursive itself. This contradicts the choice of M by (5).

Now suppose that $\neg Eij$. We claim that $X = P - P_i$ and $Y = Q - P_j$ form a counterexample to $\psi(P, Q; \overline{D})$. As $\neg Eij$, a given $M \in \mathbf{M}_0$ satisfies $M \cap P_i \in J_0$ or $M \cap P_j \in J_1$, say the first. Then, since $M \cap P \subseteq (M \cap (P - P_i)) \cup (M \cap P_i)$, $M \cap P - X \subseteq M \cap P_i \in J_0$. □

Since the Ideal Definability Lemma relativizes to any p.o. $\mathcal{E}^Z$, the previous coding results also relativize. In the relativized versions, the notions "u.r.e.", "recursive", "$\emptyset^{(c)}$" and "Σ^0_c" have to be replaced by "u.r.e. in Z", "recursive in Z", "$Z^{(c)}$" and "$\Sigma^0_c(Z)$", respectively.

3. The theories of relativized versions of $\mathcal{E}$.

We use the results in Section 2 to reprove Harrington's result that true arithmetic can be interpreted in $\mathrm{Th}(\mathcal{E})$ (assuming the Ideal Definability Lemma). More general, we prove that, if Z is implicitly definable in arithmetic, then $\mathrm{Th}(\omega, +, \times, Z)$ can be interpreted in $\mathrm{Th}(\mathcal{E}^Z)$. Since an interpretation in the other direction exists as well, the two theories have the same m-degree. Here Z is called implicitly definable in arithmetic if there is a formula ψ_Z in the language $L(+, \times)$ extended by a unary predicate R such that, for each $X \subseteq \omega$,

$$(\omega, +, \times) \models \psi_Z(X) \Leftrightarrow X = Z. \tag{9}$$

Note a set which is implicitly definable in arithmetic is hyperarithmetical and that implicit definability of Z only depends on the arithmetical degree of Z. Hence each Z which is in the same arithmetical degree as some $\emptyset^{(\alpha)}$, α a recursive ordinal, is implicitly definable in arithmetic. However, "most" hyperarithmetical sets are not implicitly definable in arithmetic, since both arithmetically generic sets and arithmetically random sets Z cannot be implicitly definable (see [N 95]).

We exploit the coding power of a specific collection of formulas in $\mathcal{E}^Z$ to show that for some fixed $c \in \omega$, if Z is implicitly definable in arithmetic and $Z^{(c)} \neq W^{(c)}$, then $\mathcal{E}^Z$ is not elementarily equivalent to $\mathcal{E}^W$. (In [S 81], similar questions were considered for relativizations of the structure of Δ^0_2 Turing-degrees.) In particular, if $Z = \emptyset^{(\alpha)}$, $W = \emptyset^{(\beta)}$, $\beta < \alpha$ recursive ordinals, then $\mathcal{E}^Z \not\equiv \mathcal{E}^W$. For finite α, β, this gives a negative answer to the question of E. Herrmann mentioned in Section

1. As a further application, if Z is sufficiently complex, namely $Z \notin Low_c$, then $\mathcal{E}^Z$ is not elementarily equivalent to $\mathcal{E}$. This includes the case that Z is arithmetically generic. We note that, for all arithmetically generic Z, the relativization $\mathcal{E}^Z$ has the same theory. Similar remarks apply to arithmetically random sets.

We begin with the relevant framework, describing how the coding results in the previous section determine a scheme of formulas such that, with appropriate parameters, a standard model of arithmetic is coded. Note that some of the conditions required for a coding configuration cannot be expressed in first order logic, so we have to be more general in our framework. Let A be an r.e. non-recursive set and I be a p-acceptable ideal, where p is as in (a) of Lemma 2.2. We think of I as defined by the appropriate formula determined in the Ideal Definability Lemma from parameters in a list $\overline{P}$ including A. Then the formula $\varphi_U(X;\overline{P})$ defining the universe of the structure to be coded is a formula expressing

"X/I is an atom in $\mathcal{B}(A)/I$",

and equality of the structure is defined in $\mathcal{E}$ by a formula $\varphi_\sim(X,Y;\overline{P})$ expressing "$X/I = Y/I$" ($X,Y \in \mathcal{B}(A)$). Finally, from the Coding Lemma for ternary recursive relations, we obtain formulas $\varphi_+(X,Y,Z;\overline{P})$ and $\varphi_\times(X,Y,Z;\overline{P})$ intended to code the arithmetical operations on atoms P/I. We assume that $\overline{P}$ includes all the parameters needed. Lemma 2.2 and the Coding Lemma 2.3 show that, for some special list $\overline{P}$, a standard model of arithmetic is coded.

To give an interpretation of true arithmetic in Th($\mathcal{E}$), it now suffices to give a first-order condition on $\overline{P}$ which is shared by such a special list and always implies that the model coded is standard. As an aid, we first require the following "correctness conditions" (which can be formulated as first-order conditions on the parameters):

- φ_+ and $\varphi_\times$ determine binary total functions on $\{X/I : \varphi_U(X;\overline{P})\}$.
- The structure for $L(+,\times)$ coded by $\overline{P}$ satisfies a sufficiently large fragment PA^- of Peano arithmetic which implies that the structure has an initial segment isomorphic to ω.

We use variables $M, M_0, \ldots$ to denote structures for $L(+,\times)$ coded in $\mathcal{E}$ which satisfy the correctness conditions. If we need to refer to the list of parameters $\overline{P}$ involved explicitly, we write $M(\overline{P})$. Moreover, if $i \in \omega$, we write i^M for the standard number i in M. The variables P, Q range over $\{X : X/I \text{ atom in } \mathcal{B}(A)/I\}$. By the relativizability of the results in Section 2, the same scheme works in $\mathcal{E}^Z$. We make some observations which will enable us to interpret true arithmetic in Th($\mathcal{E}^Z$) for each Z and Th$(\omega,+,\times,Z)$ in Th($\mathcal{E}^Z$) if Z is implicitly definable in arithmetic.

(10) If $\varphi(\overline{X};\overline{P})$ is a Σ^0_k formula with parameters in the language of $\mathcal{E}$, then for each Z, the index set with respect to the indexing of $\mathcal{E}^Z$, $(W^Z_e)_{e\in\omega}$, of the relation defined by φ with a fixed parameter list is recursive in $Z^{(k+2)}$.

(11) For some fixed number h (which does not depend on Z), for each M, there is $g \leq_T Z^{(h)}$ such that

$$(\forall i)[(W^Z_{g(i)})/I = i^M].$$

Proof. (10) is immediate since "$W^Z_i \subseteq W^Z_j$" is recursive in $Z^{(2)}$. For (11), suppose that $M = M(\overline{P})$. Let $\varphi_S(X,Y;\overline{P})$ be a formula defining the successor function in (any) $M(\overline{P})$. By (10), the corresponding binary relation on indices is recursive in

$Z^{(h)}$ for some fixed number h, so there is a partial "choice" map f which can be computed with the oracle $Z^{(h)}$ such that, in $\mathcal{E}^Z$,

$$\varphi_S(W_i^Z, W_j^Z; \overline{P}) \text{ for some } j \Rightarrow \varphi_S(W_i^Z, W_{f(i)}^Z; \overline{P}).$$

Fix i_0 such that $W_{i_0}^Z/I = 0^M$. Then, by iterating f with i_0 as an initial value, obtain g as desired. The fact (11) immediately implies

(12) For each M, $\{e : W_e^Z/I$ is a standard number of $M\}$ is $\Sigma_p^0(Z)$ for some fixed p.

□

3.1 Theorem. *If Z is implicitly definable in arithmetic, then there are interpretations of theories which show $Th(\mathcal{E}^Z) \equiv_m Th(\omega, +, \times, Z)$.*

Proof. Clearly, $\mathrm{Th}(\mathcal{E}^Z)$ can be interpreted in $\mathrm{Th}(\omega, +, \times, Z)$. For the other direction, we first give another proof of Harrington's result that the theorem holds with $Z = \emptyset$. Suppose M is given and $S \subseteq M$. Since atoms of a Boolean algebra (here: $\mathcal{B}(A)/I$) are independent (i.e. no atom is below a finite sup of other atoms), if I_S is the ideal of $\mathcal{B}(A)$ generated by $\{P : P/I \in S\} \cup I$, then "$I_S \cap M = S$", i.e. for each P,

$$P/I \in S \Leftrightarrow P \in I_S. \tag{13}$$

(Forming the "intersection" $J \cap M$ above makes sense for any ideal J which contains I.) Moreover, if $\{e : W_e/I \in S\}$ is Σ_k^0, then for sufficiently large k, I_S has Σ_k^0 index set and hence is k-acceptable. Thus we can use the Ideal Definability Lemma to quantify over a class of subsets of M which contains the Σ_k^0 subsets. By (12), the standard part of M is such a set for appropriate k. Therefore the following holds iff M is standard, and can be expressed as a first-order condition on the list of parameters coding M:

> *"each subset S of M such that I_S is k-acceptable which is closed under successor and contains 0^M equals M".*

This gives an interpretation of true arithmetic in $\mathrm{Th}(\mathcal{E})$. By relativization, for each Z, we can express if M coded in $\mathcal{E}^Z$ is standard, so we also obtain an interpretation in $\mathrm{Th}(\mathcal{E}^Z)$.

Now suppose Z is implicitly definable in arithmetic. To interpret $\mathrm{Th}(N, +, \times, Z)$ in $\mathrm{Th}(\mathcal{E}^Z)$ we need an extended scheme which enables us to encode structures $(M, \hat{Z})$, where M is a standard model of arithmetic and $\hat{Z}$ is Z, viewed as a subset of M. Let ψ_Z be a formula describing Z as in (9). Given M as above, let $I_{\hat{Z}}$ be the ideal of $\mathcal{B}(A)$ generated by I and $\{P : P/I \in \hat{Z}\}$. Then, using the map g from (11), $I_{\hat{Z}}$ equals the ideal generated by $\{W_{g(n)}^Z : n \in Z\} \cup I$. Since $g \leq_T Z^{(h)}$ for some h, $I_{\hat{Z}}$ is p-acceptable (in $\mathcal{E}^Z$) for some p.

In the extended scheme, expand the list of parameters by parameters defining a p-acceptable ideal J of $\mathcal{B}(A)$. Require as a correctness condition on the scheme that $I \subseteq J$. From J, define a subset S of M by

$$P/I \in X \Leftrightarrow P \in J$$

(the intended meaning is $S = \hat{Z}$). Suppose M is standard. By the independence argument in (13), $P/I \in \hat{Z} \Leftrightarrow P \in I_{\hat{Z}}$. The interpretation of $\mathrm{Th}(\omega, +, \times, Z)$ is now given by $(\omega, +, \times, Z) \models \varphi \Leftrightarrow$ for some (M, X) defined by the extended scheme, M is standard,X (as a subset of M) satisfies the description of S and $(M, X) \models \varphi$ (this can be expressed in first-order logic). □

3.2 Theorem. *There exists a number $c \in \omega$ such that, if $Z^{(c)} \not\equiv_T W^{(c)}$ and Z or W is implicitly definable in arithmetic, then $\mathcal{E}^Z \not\equiv \mathcal{E}^W$.*

3.3 Corollary. *If α is a recursive ordinal and $\beta < \alpha$, then $\mathcal{E}^{\emptyset^{(\alpha)}} \not\equiv \mathcal{E}^{\emptyset^{(\beta)}}$.* □

Proof of the Theorem. Let q be a number such that for each M coded in $\mathcal{E}^X$ the associated ideal I is q-acceptable and, by (11), there exists a map $g \leq_T X^{(q)}$ such that

$$(\forall n)[W^X_{g(n)}/I = n^M].$$

Fix M coded in $\mathcal{E}^X$. We first make the following observation. Let $p > q$, $S \subseteq \omega$, and let I_S be the ideal of $\mathcal{B}(A)$ generated by $\{W^X_{g(n)} : n \in S\} \cup I$. Then

$$S \text{ is } \Sigma^0_p(X) \Leftrightarrow \{e : W^X_e \in I_S\} \text{ is } \Sigma^0_p(X). \tag{14}$$

For the direction from left to right, note that

$$\begin{aligned} W^X_e \in I_S \Leftrightarrow & (\exists F \subset \omega \text{ finite })(\exists e_0) \\ & [F \subseteq S \wedge W^X_{e_0} \in I \wedge \\ & \quad W^X_e \subseteq \bigcup_{i \in F} W^X_{g(i)} \cup W^X_{e_0}]. \end{aligned}$$

It is easy to check that this can be expressed as a $\Sigma^0_p(X)$ property of e. For the other direction, if I_S has a $\Sigma^0_p(X)$ index set, then, because

$$\begin{aligned} n \in S & \Leftrightarrow W^X_{g(n)} \in I_S \\ & \Leftrightarrow (\exists e)[g(n) = e \wedge W^X_e \in I_S] \end{aligned}$$

and $q < p$, S is $\Sigma^0_p(X)$.

Now let $c > q$ be even and let $p = c + 1$. Assume $Z^{(c)} \not\leq_T W^{(c)}$. Then, if $S = Z^{(p)}$, $S \in \Sigma^0_p(Z)–\Sigma^0_p(W)$. Let $\varphi(Y; A, \overline{C})$ be the formula obtained from the Ideal Definability Lemma to define uniformly in $\mathcal{E}^X$ for a set A which is r.e., but not recursive in X all p-acceptable ideals of $\mathcal{B}(A)$.

First suppose that Z is implicitly definable in arithmetic via the description ψ_Z. Then the following is true in $\mathcal{E}^Z$, but not in $\mathcal{E}^W$.

> There is a structure (M, Y), coded by the extended scheme, such that M is standard, $(M, Y) \models \psi_Z(Y)$ and, for some list $\overline{C}$, the intersection of M and the ideal coded by $A, \overline{C}$ equals $Y^{(p)}$, i.e.

$$(\forall P)[(M, Y) \models P/I \in Y^{(p)} \Leftrightarrow \varphi(P; \overline{C})]. \tag{15}$$

The statement holds in $\mathcal{E}^Z$ via any standard M and $Y = \hat{Z}$ (i.e. Z viewed as a subset of M), for in this case $I_{Z^{(p)}}$ is p-acceptable. In $\mathcal{E}^W$, either in no structure

(M, Y) defined by the extended scheme does $\psi_Z(Y)$ hold, or, if (M, Y) is such a structure, then (15) fails. For, in $\mathcal{E}^W$, $\{P \in \mathcal{B}(A) : \varphi(P, \overline{C})\}$ is an ideal with $\Sigma^0_p(W)$ index set by the strictness of the Ideal Definability Lemma. So, by (14), $Z^{(p)} \in \Sigma^0_p(W)$, a contradiction.

Now suppose W is implicitly definable. The case that $W^{(c)} \not\leq_T Z^{(c)}$ is already covered above. Otherwise there is an index e such that $\{e\}^{Z^{(c)}} = W$. Then the following is true in $\mathcal{E}^Z$, but not in $\mathcal{E}^W$.

> There is a coded standard model M and a list $\overline{C}$ coding an ideal of $\mathcal{B}(A)$ which contains I such that if U is the intersection of M and the ideal coded, then for some index $e \in M$, $\{e\}^U$ satisfies the description of W and U is not in $\Sigma^0_p(\{e\}^U)$.

This statement holds in $\mathcal{E}^Z$ via the ideal $I_{Z^{(p)}}$, but fails in $\mathcal{E}^W$, again by the strictness of the Ideal Definability Lemma. □

4. Definability of classes of hyperhypersimple sets in $\mathcal{E}$.

We give two different ways to define quasimaximality. In the first, the first-order definition is obtained as follows: A is quasimaximal iff $\mathcal{L}^*(A)$ is an atomic Boolean algebra, but it is not possible to code a successor model using the formulas arising from the above coding configuration. To verify this, it is shown that a coding of a successor model would require the existence of infinitely many atoms in $\mathcal{L}^*(A)$. Here a successor model is a structure with a binary relation E which is a $1-1$ but not onto map from the universe of the structure into itself. Clearly the universe of such a structure must be infinite. Let p be a number such that in Lemma 2.2, I is p-acceptable and, if J_0, J_1 code a recursive binary relation as in the Coding Lemma 2.3, the ideals J_0, J_1 are p-acceptable. Since A is quasimaximal iff $\mathcal{L}^*(A)$ is a finite Boolean algebra, it suffices to prove the following.

4.1 Lemma. *Suppose $\mathcal{L}^*(A)$ is an atomic Boolean algebra. Then A is not quasimaximal iff*

(16) *there is a p-acceptable ideal I of $\mathcal{B}(A)$ and there are p-acceptable ideals J_0, J_1 of $\mathcal{B}(A)$ such that J_0, J_1 code a successor model on the atoms of $\mathcal{B}(A)/I$ via* (8).

4.2 Theorem. *The class of quasimaximal sets is definable in $\mathcal{E}$.*

Proof. Clearly we can express in first-order logic that $\mathcal{L}^*(A)$ is an atomic Boolean algebra. Furthermore, we can express (16) by the Ideal Definability Lemma. □

Proof of Lemma 4.1. If A is not quasimaximal then $\mathcal{L}^*(A)$ possesses infinitely many atoms, so by Lemma 2.2 there is a coding configuration based on A. Then the successor model where P_{k+1}/I is the successor of P_k/I can be coded by p-acceptable ideals of $\mathcal{B}(A)$.

Now suppose for a contradiction A is quasimaximal, but (16) is satisfied via I, J_0 and J_1. Choose recursive sets $M_0, \ldots, M_{r-1}$ such that each atom in $\mathcal{L}^*(A)$ is of the form $(A \cup M_i)^*$ for some i. Moreover, choose a sequence of representatives $Q_k \sqsubset A$ such that for each $k \in \omega$, Q_k/I is the standard element in the coded successor model corresponding to k (here first we fix some "zero" element Q_0/I which is not in the range of the map coded by J_0, J_1 on the atoms of $\mathcal{B}(A)/I$). By (8) for each

$u, v \in \omega, u + 1 = v$ iff

$$(17) \qquad (\forall X, Y \in I)(\exists i < r) \; [M_i \cap Q_u - X \notin J_0 \wedge M_i \cap Q_v - Y \notin J_1].$$

We claim that one can exchange the quantifiers in the expression above in case $u + 1 = v$. Fix a generating sequence (X_k) for I such that $X_k \subseteq X_{k+1}$ for each k. Since (17) is closed downwards in X and Y, it suffices to require (17) for each k and $X = Y = X_k$. For each u there is $i(u) < r$ such that for infinitely many k, $M_{i(u)} \cap P_u - X_k \notin J_0$ and $M_{i(u)} \cap P_{u+1} - X_k \notin J_1$. So if $u + 1 = v$, for each $X, Y \in I$ (17) holds via $i = i(u)$. Now choose $u, u', u+1 < u'$, such that $i(u) = i(u')$. Then (17) shows that the successor relation holds between P_u/I and $P_{u'+1}/I$, i.e. $u + 1 = u' + 1$, a contradiction. □

We now consider definability of classes of hh-simple sets based on the Ideal Definability Lemma alone, thereby giving an alternative way to define quasimaximality. We need two facts.

Fact 1. *If $\mathcal{L}^*(A)$ is a boolean algebra, then there is a $\emptyset''$– isomorphism $\Theta : \mathcal{L}^*(A) \to \mathcal{B}^*_A$, where $\mathcal{B}^*_A = \{(R \cap A) : R$ recursive$\}$.*

(Proof: Let $\Theta(B^*) = (R \cap A)^*$, where $B = A \cup R$. Note that it takes an oracle $\emptyset''$ to find R, from e such that $B = W_e$.) Observe that $\mathcal{B}^*_A$ is a subalgebra of $\mathcal{B}^*(A)$ containing $\mathcal{R}^*(A)$. Thus we also obtain an isomorphism of the lattice of Σ^0_k-ideals I of $\mathcal{L}^*(A)$ $(k \geq 3)$ onto the lattice of Σ^0_k-ideals $\tilde{I}$ of $\mathcal{B}^*_A$ which contain $\mathcal{R}^*(A)$. The Ideal Definability Lemma now implies that the Σ^0_k-ideals of $\mathcal{L}^*(A)$ $(k \geq 3$ odd) are uniformly definable, because $\tilde{I} = [\tilde{I}]_{\mathrm{id}} \cap \mathcal{B}^*_A$, where $[\tilde{I}]_{\mathrm{id}}$ is the (k-acceptable) ideal of $\mathcal{B}^*(A)$ generated by $\tilde{I}$.

Fix a *hh*-simple A as a parameter. We consider definability of ideals of $\mathcal{L}^*(A)$ with parameter A^* in $\mathcal{E}^*$. If I is an ideal of $\mathcal{L}^*(A)$, let $\mathcal{A}(I)$ be the ideal of $\mathcal{L}^*(A)$ generated by the atoms of $\mathcal{L}^*(A)/I$ (i.e., $\mathcal{L}^*(A)/\mathcal{A}(I)$ is the derivative of $\mathcal{L}^*(A)/I$).

Fact 2. *If I is an ideal of $\mathcal{L}^*(A)$ which is definable in $(\mathcal{E}^*, A^*)$, then so is $\mathcal{A}(I)$. The formula defining $\mathcal{A}(I)$ only depends on the formula defining I, not on A.*

Proof. If I in a Σ^0_k ideal $(k \geq 3)$, then $\mathcal{A}(I)$ must be Σ^0_{k+2}. So we can define $\mathcal{A}(I)$ as the least Σ^0_{k+2} ideal of $\mathcal{L}^*(A)$ which contains all elements of I and all $B^* \geq A^*$ such that B^*/I is an atom in $\mathcal{L}^*(A)/I$. This proves Fact 2.

Note that we can also express if $\mathcal{A}(I)$ contains infinitely many atoms of $\mathcal{L}^*(A)/I$: this is the case iff $\mathcal{A}(I)$ describes a nonprincipal ideal in $\mathcal{L}^*(A)/I$, i.e. if there is no $B^* \geq A^*$ such that, for each $C^* \supseteq A^*$, $C^* \in \mathcal{A}(I) \Leftrightarrow (C - B)^* \in I$.

In the following Theorem, (i) for $n = 1$ gives an alternative first–order definition of quasimaximality. In (ii), we refer to Ershov's classification of the completions of the theory of Boolean algebras, as presented in [CK 90], Section 5.5.

4.3 Theorem. *The following classes of hh-simple sets are definable.*

(i) $\{A :$ *the n-th derivative of* $\mathcal{L}^*(A)$ *is* $\{0\}\}$

(ii) $\{A : \mathcal{L}^*(A) \models T\}$, *where T is any completion of the theory of BA's except the one with the characteristic $m(T) = \infty$ in the notation of [CK90].*

Proof.

(i) Let I_0^A be the least ideal of $\mathcal{L}^*(A)$, and for each n let $I_{n+1}^A = \mathcal{A}(I_n^A)$. Then, by Fact 2, there is a formula φ_n which uniformly for each A^* defines I_n^A in $\mathcal{L}^*(A)$. So we can express that $I_n^A = \mathcal{L}^*(A)$.

(ii) is left to the reader.

5. Noncoding Theorems.

Let the variable $\mathbf{R}$ range over finite classes of pairwise disjoint infinite recursive sets. We use the variable $\tilde{X}$ for tuples of recursive sets $(X_0, \ldots, X_{n-1})$.

5.1 Theorem. *For each formula $\varphi(X, Y; \tilde{P})$ a number k can be found effectively such that for each $\mathbf{R}$ such that $|\mathbf{R}| \geq k$ and for each list of parameters $\tilde{A}$, the relation*

$$\{(X, Y) : X, Y \in \mathbf{R} \wedge \mathcal{E} \models \varphi(X, Y; \tilde{A})\}$$

is not a linear ordering of $\mathbf{R}$.

5.2 Corollary. *If $\mathcal{L}^*(A)$ is a Boolean algebra with infinitely many atoms, then it is not possible to define, even with parameters, a linear ordering on the atoms of $\mathcal{L}^*(A)$.*

Proof of the Corollary. If F is a set of atoms and $|F| = k$, then for some $\mathbf{R}$, $|\mathbf{R}| = k$, $F = \{A \cup R^* : R \in \mathbf{R}\}$. Hence, if $\psi(X, Y; \tilde{P})$ defines a linear order on the atoms, then $\varphi(X, Y; \tilde{P}, A) \equiv \psi(X \cup A, Y \cup A; \tilde{P})$ defines a linear order on sets $\mathbf{R}$ of arbitrarily large cardinality.

Proof of the Theorem. Note that, if R, S and $T = \overline{R \cup S}$ are infinite, then $\mathcal{E} \cong \mathcal{E}^3$ via the map

$$X \mapsto (X \cap R, X \cap S, X \cap T).$$

By a result of Feferman and Vaught [FV59], if $\mathbf{A}$ is a structure and $\varphi(X^0, \ldots, X^{n-1})$ is a formula in the language of $\mathbf{A}$, then

$$\mathbf{A}^3 \models \varphi \begin{pmatrix} a_0^0 & & a_0^{n-1} \\ a_1^0 & ,\ldots, & a_1^{n-1} \\ a_2^0 & & a_2^{n-1} \end{pmatrix} \Leftrightarrow$$
$$\bigvee_{\alpha=1,\ldots,r} \bigwedge_{i=0,1,2} \mathbf{A} \models \varphi_i^\alpha \left(a_i^0, \ldots, a_i^{n-1}\right)$$

for some formulas φ_i^α which only depend on φ and can be found effectively (this can be proved by induction on φ). Now suppose $\varphi(X, Y; \tilde{P})$ defines a linear order $<_L$ on a set $\mathbf{R}$. By the isomorphisms $\mathcal{E} \Leftrightarrow \mathcal{E}^3$ above, an element $A \in \mathcal{E}$ corresponds to the vector

$$\begin{pmatrix} A \cap R \\ A \cap S \\ A \cap T \end{pmatrix}.$$

Hence, if $R, S \in \mathbf{R}, R \neq S$, then

$$R <_L S \Leftrightarrow \bigvee_{\alpha=1,\ldots,r} (\mathcal{E}(R) \models \varphi_0^\alpha(R, \emptyset, \tilde{P} \cap R) \wedge$$
$$\mathcal{E}(S) \models \varphi_1^\alpha(\emptyset, S, \tilde{P} \cap S) \wedge$$
$$\mathcal{E}(T) \models \varphi_2^\alpha(\emptyset, \emptyset, \tilde{P} \cap T)),$$

where $T = \overline{R \cup S}$ and $\tilde{P} \cap X = (P_0 \cap X, \ldots, P_{k-1} \cap X)$. Note that "$\mathcal{E}(T) \models \varphi_2^\alpha(\emptyset, \emptyset, \tilde{P} \cap T)$" does not depend on the order of R, S. We say that $R <_L S$ via α if the disjunct corresponding to α holds. Now, we can compute a number M such that, for $|\mathbf{R}| \geq M$, there exist α and $A, B, C, D \in \mathbf{R}$ such that $A <_L B <_L C <_L D$ and the ordering relations hold all via α. This is verified by using Ramsey's Theorem: assign one of r possible colors to $\{X, Y\} \subseteq \mathbf{R}, X \neq Y$, according to the minimum $\alpha \leq r$ such that $X <_L Y$ or $Y <_L X$ holds via α. For $k = |\mathbf{R}|$ large enough, there exists a homogeneous set F for this coloring of cardinality 4. Since either $X <_L Y$ or $Y <_L X$ for each $X, Y \in \mathbf{R}, X \neq Y$, there must be α such that, for $X, Y \in F$,

$$X <_L Y \Leftrightarrow X <_L Y \quad \text{via} \quad \alpha.$$

Now let $F = \{A, B, C, D\}, A <_L B <_L C <_L D$. We show $C <_L B$, a contradiction. $\mathcal{E}(C) \models \varphi_0^\alpha(C, \emptyset, \tilde{P} \cap C)$ holds since $C <_L D$ via α, and $\mathcal{E}(B) \models \varphi_1^\alpha(\emptyset, B, \tilde{P} \cap B)$ because $A <_L B$ via α. Finally $\mathcal{E}(\overline{B \cup C}) \models \varphi_2^\alpha(\emptyset, \emptyset, \tilde{P} \cap (\overline{B \cup C}))$ is true since $B <_L C$. This shows $C <_L B$ via α. □

5.3 Theorem [Harrington]. *(see also* [Ho,N ta]*) It is not possible to code an infinite linear ordering in $\mathcal{E}$ without parameters.*

Proof. Suppose for a contradiction that there is an $\mathcal{E}$-definable $2n$–ary relation $\leq_L$ which is a linear preordering on $\mathcal{E}^n$ such that the equivalence relation $\tilde{X} \equiv_L \tilde{Y} \Leftrightarrow \tilde{X} \leq_L \tilde{Y} \leq_L \tilde{X}$ has infinitely many equivalence classes. We say that a recursive set R supports A if $A \subseteq R$ or $\overline{R} \subseteq A$. R supports $(A_0, \ldots, A_{n-1})$ if R supports each set A_i. Let $\mathbf{C} = \{R : |R| = |\overline{R}| = \infty\}$.

5.3 Lemma. *For each tuple $\tilde{A} = (A_0, \ldots, A_{n-1})$ of sets there exists $R \in \mathbf{C}$ such that R supports $\tilde{A}$.*

Proof. We say that S co-supports A if $\overline{S}$ supports A, i.e. $S \subseteq A$ or $A \subseteq \overline{S}$. This notion is closed downwards in S. We define inductively sets $S_k \in \mathbf{C}$ co– supporting $A_0, \ldots, A_k$. Then $R = \overline{S}_{n-1}$ is as required.

Let S_0 be a set in $\mathbf{C}$ which is a subset of A_0 if A_0 is infinite and of $\overline{A}_0$ else. If $k < n-1$ and $S_k \cap A_{k+1}$ is infinite let $S_{k+1} \in \mathbf{C}$ be a recursive subset of $S_k \cap A_{k+1}$. Else let $S_{k+1} = S_k - A_{k+1}$. □

We now derive an effective bound on $|\mathcal{E}^n / \equiv_L|$ (depending on the defining formula for $\leq_L$). First we show that each equivalence class of $\equiv_L$ is large in the following sense: for each $\tilde{A} \in \mathcal{E}^n$,

$$(\forall S \in \mathbf{C})(\exists \tilde{B} \equiv_L \tilde{A})[S \text{ supports } \tilde{B}] \tag{18}$$

Fix $R \in \mathbf{C}$ supporting $\tilde{A}$, and let $S \in \mathbf{C}$ be arbitrary. First suppose that $R \cap S = \emptyset$, and let π be a recursive permutation of order 2 which exchanges R and S and is the identity on $\overline{R \cup S}$. Let $B_i = \pi(A_i)(i < n)$. Then S supports $\tilde{B}$. Now $\tilde{A} \leq_L \tilde{B}$ is equivalent to $\tilde{B} = \pi(\tilde{A}) \leq_L \pi(\tilde{B}) = \tilde{A}$, since $\leq_L$ is definable. So $\tilde{A} \equiv_L \tilde{B}$.
If $R \cap S$ is finite, proceed as above, replacing S by $S - R$. Then $\tilde{B}$ is supported by $S - R$ and hence by S. If $R \cap S$ is infinite, obtain first $\tilde{B}_0 \equiv_L \tilde{A}$ supported by $\overline{R}$ and then $\tilde{B} \equiv_L \tilde{B}_0$ supported by $R \cap S$. Then $\tilde{B} \equiv_L \tilde{A}$ and $\tilde{B}$ is supported by S.

Suppose $|\mathcal{E}^n / \equiv_L| \geq p$. We derive a bound on p. By (18), let $S_0, \cdots, S_{p-1} \in \mathbf{C}$ be pairwise disjoint sets and let $\tilde{B}^i$, $i < p$, be n-tuples of recursive set supported by S_i such that

$$\tilde{B}^0 <_L \cdots <_L \tilde{B}^{p-1}.$$

If a tuple $\tilde{X} = (X_0, \ldots, X_{n-1})$ is supported by S, we assign a signature $\beta \in \{0,1\}^n$ to $(\tilde{X}, S)$ by $\beta(k) = 0 \Leftrightarrow X_k \subseteq S \quad (k < n)$. Fix an arbitrary number q. If $p \geq 2^n q$, then there is a subsequence $(\tilde{A}^j, R_j)_{j<q}$ of $(\tilde{B}^i, S_i)_{i<p}$ such that all $(\tilde{A}^i, R_i)$ have the same signature β. Let

$$A_k = \bigcup_{j<q} (A_k^j \cap R_j) \qquad (k < n).$$

We show that the parameters $A_0, \ldots, A_{n-1}$ can be used to define in a first-order way a linear order on $\{R_0, \ldots, R_{q-1}\}$. Clearly one can decode each $\tilde{A}^j$ in a uniform first-order way from R_j and this list of parameters, because $A_k^j = A_k \cap R_j$ if $\beta(k) = 0$ and $A_k^j = (A_k \cap R_j) \cup \overline{R}_j$ if $\beta(k) = 1$. Thus for the formula $\psi(R, S; A_0, \ldots, A_{n-1})$ expressing $\tilde{C} <_L \tilde{D}$, where C_k is $A_k \cap R$ if $\beta(k) = 0$ and $(A_k \cap R) \cup \overline{R}$ else, and D_k is $A_k \cap S$ if $\beta(k) = 0$ and $(A_k \cap S) \cup \overline{S}$ else,

$$\psi(R_i, R_j; A_0, \ldots, A_{n-1}) \Leftrightarrow \tilde{A}^i <_L \tilde{A}^j,$$

so ψ defines a linear order on $\{R_0, \cdots, R_{q-1}\}$ with the parameters $A_0, \cdots, A_{n-1}$. By Theorem 5.1, this gives an effective bound on q depending on ψ (where ψ was obtained in an effective way from ϕ and β, but did not depend on q). Hence $|\mathcal{E}^n / \equiv_L|$ cannot exceed 2^n times this bound. Since we can take the maximum over all possible β, we effectively obtain a bound which only depends on ϕ. $\square$

5.5 Corollary. *The following relations are not definable in $\mathcal{E}$*

(i) $\{\langle A, B\rangle : n(A) \leq n(B)\}$
(ii) $\{(A, B) : |A| \leq |B|\}$

Proof. Definability would enable us to code $(\omega, \leq)$ on equivalence classes. $\square$

Let $A \approx B$ denote that A is automorphic to B in $\mathcal{E}$. R. Soare [So 74] proves that, for quasimaximal A, B,

$$A \approx B \Leftrightarrow n(A) = n(B).$$

Therefore,

$$n(A) \leq n(B) \Leftrightarrow (\exists B')[B \subseteq B' \wedge B' \approx A].$$

In fact the automorphism involved can be represented by a Σ_3^0 map on indices.

5.6 Corollary. *The following relations (which are $=^*$ invariant) are non–definable in $\mathcal{E}$:*

(i) $A \approx B$
(ii) $A \approx B$ *via a* Σ_3^0 *automorphism*
(iii) $\mathcal{L}^*(A) \cong \mathcal{L}^*(B)$

Proof:. Definability of either one of the relations, together with Theorem 4.2, would imply the definability of

$$\{(A, B) : A, B \quad \text{quasimaximal} \wedge \quad n(A) \leq n(B)\},$$

so $(\omega, \leq)$ could be coded in $\mathcal{E}$ without parameters. $\square$

6. Appendix: Proof of Harrington's Ideal Definability Lemma.

Recall that, for $n = 1$,

$$\varphi_1(X; A, C) \equiv X \sqsubset A \wedge (\exists R \subseteq A)[\quad X \subseteq C \cup R]$$

and that, if $\overline{C} = (C_0, \dots, C_{n-1})$,

$$\varphi_{n+1}(X; A, \overline{C}, C_n) \equiv X \sqsubset A \wedge$$
$$(\exists R \subseteq A\ rec)(\forall S \subseteq A - R\ rec)$$
$$\varphi_n(X \cap S \cap C_n; C_n, \overline{C}).$$

Lemma 1. *If A is r.e. and I is a 3-acceptable ideal of $\mathcal{B}(A)$, then there is $C \subseteq A$ such that for $X \in \mathcal{B}(A)$,*
$X \in I \Leftrightarrow X \subseteq C \cup R$ for some recursive $R \subseteq A$.
Hence φ_1 defines precisely the 3-acceptable ideals.

Proof of Lemma 1.

Step 1. Uniformly in an index of a subset Z of A we can obtain a pair of disjoint sets S, T such that

- $S \subseteq A \subseteq S \cup T$
- $Z = A \Rightarrow S \cup T = \omega$
- $Z \neq A \Rightarrow T$ finite.

(So, if $Z \neq A$, $S =^* A$.)
Proof: Let

$$S = \{a : (\exists s)[a \in A_s \wedge A_a \not\subseteq Z_s]\}$$
$$T = \{b : (\exists s)[A_b \subseteq Z_s \wedge b \notin A_{s-1}]\}$$

(Intuitively speaking, S contains those elements which are enumerated late, and T those which are enumerated early.) We verify the required properties of S, T. For instance, to show $S \cap T = \emptyset$, suppose $b \in T$ via a witness s. If $b \in A_t$, where $t \geq s$ is minimal, then $A_b \subseteq Z_s \subseteq Z_t$, therefore $b \notin S$. The verifications of the other properties are left to the reader.

Step 2. If a class H of splitting components of A is given s.t. "$W_i \in H$" is Σ^0_3, then clearly, one can obtain u.r.e. sequences (X_i, Y_i) of pairs of disjoint subsets of A such that $W_e \in H \Leftrightarrow \exists i\ W_e = X_i$, and $X_i \cup Y_i = A$ or $X_i \cup Y_i$ is finite.

Apply this to $H = I$ to obtain the sequences (X_i), (Y_i), and also to $H = \mathcal{B}(A)$ to obtain (U_i), (V_i). Now apply Step 1 to $Z_i = X_i \cup Y_i$ to obtain sequences (S_i, T_i), and to $Z_i = U_i \cup V_i$ to obtain $(\tilde{S}_i, \tilde{T}_i)$. Note that

$$A =^* (X_i \cup S_i) \cup (Y_i \cap T_i)$$

and also

$$A =^* (V_j \cup \tilde{S}_j) \cup (U_j \cap \tilde{T}_j)$$

since $S_i =^* A$ if $X_i \cup Y_i$ is finite and $S_i \cup T_i = \omega$ else. Thus, at the cost of changing X_i and Y_i on a recursive subset of A, we have achieved that that their union *always is almost equal to* A.

Observe that, in the above, V_j *(not* U_j*) plays the role similar to the role of* X_i.

Step 3. Let

$$P^j_s = ((V_j \cup \tilde{S}_j) \cup (U_j \cap \tilde{T}_j))[s] \cap \bigcap_{i \leq j} (X_i \cup S_i \cup (Y_i \cap T_i))[s]$$

Note that $P^j =^* A$. $A - P_j$ *can be viewed as a set of "exceptions".*

Let $\sigma^j_s(a) = \{k \leq j : a \in X_{k,s}\}$ *and* $\rho^j_s(a) = \{k \leq j : a \in Y_k \cap T_k\}$ *be the* j*-states of* A *w.r.t.* (X_k) *and* $(Y_k \cap T_k)$.

Step 4. Now let $C_0 = \emptyset$, *and let*

$$\begin{aligned} C_{s+1} = & C_s \cup \{a : (\exists i \leq s)[a \in X_{i,s} \wedge \\ \text{(a)} \quad & (\forall j < i)[a \in P^j_s \wedge \\ \text{(b)} \quad & (a \in V_{j,s} \cup \tilde{S}_{j,s} \vee \\ \text{(c)} \quad & (\exists b < a)[b \in P^s_j \wedge b \notin C_s \wedge \\ & \sigma^j_s(a) = \sigma^j_s(b) \wedge \rho^j_s(a) = \rho^j_s(b) \wedge \\ & b \in U_{j,s} \cap \tilde{T}_{j,s}])]] \end{aligned}$$

Condition (a) only keeps finitely many elements from X_i *out of* C. *The conditions can be explained as follows: for each* $j < i$, *if (b) fails, then* $a \in U_{j,s} \cap \tilde{T}_{j,s}$ *by definition of* P^j_s, *in which case we allow* a *into* C *only if there is a* $b < a$ *with similar properties which is not in* C.

For fixed i *and each* $j < i$, *there are only finitely many such types of elements, so almost all elements of* X_i *will be enumerated into* C. *This implies that* I *is included in the ideal of elements* X *satisfying* $\varphi_1(X, A, C)$, *as we verify now.*

Claim 1. *For each* i, $X_i \subseteq^* C$.

Proof of Claim 1.. For any i, j, $\bigcap_{j<i} P^j =^* A$, so (a) only holds back finitely many elements from X_i. Moreover, for a.e. a there is $b < a$ which behaves the same way with respect to membership in finitely many given sets, so $\sigma^j(a) = \sigma^j(b)$, $\rho^j(a) = \rho^j(b)$ and $a \in U_j \cap \tilde{T}_j \leftrightarrow b \in U_j \cap \tilde{T}_j$. Thus, for all $j < i$ and a.e. a, if (b) fails, then $a \in U_j \cap \tilde{T}_j$ and (c) must hold. Thus $X_i \subseteq^* C$.

Claim 2. *Suppose* $U \sqsubset A$. *If* $U \subseteq C$, *then* $U \in I$.

Claim 1 and Claim 2 will establish the lemma: If $\varphi_3(X, A, C)$ holds then $U = X - S \subseteq C$ for some recursive $S \subseteq A$, so $X - S \in I$, so $X \in I$ since $\mathcal{R}(A) \subseteq I$.

Proof of Claim 2. Let $U = U_j$, where $U_j \cup V_j = A$. We show $U \subseteq^* X_0 \cup \cdots \cup X_j \cup R$ for some recursive $R \subseteq A$: let

$$\begin{aligned} F &= \{k \leq j : X_k \cup Y_k \neq A\}, \\ G &= \{0, \ldots, j\} - F \text{ and} \\ R &= \left(\bigcup_{k \in G} S_k \right) \cup \tilde{S}_j. \end{aligned}$$

Then $R \subseteq A$. Assume for a contradiction that $U_j \not\subseteq^* X_0 \cup \cdots \cup X_j \cup R$. We show $U_j \not\subseteq C$, contrary to our assumption.

If $k \in F$, then X_k is finite. Let t be such that $X_{k,t} = X_k$ for $k \in F$, and let $a = \min(U_j - (X_0 \cup \cdots \cup X_j \cup R \cup C_t))$. We show $a \notin C$. Else, suppose $a \in C_{s+1} - C_s$ $(s \geq t)$. Then $a \in X_{i,s}$, where $j < i$ and $i \leq s$. Since $a \in U_j - \tilde{S}_j$, the alternative (c) must hold for j via some $b < a$.

We show that b could replace a, contrary to the minimality of a. Since $b \in U_j \cap \tilde{T}_j$, $b \in U_j - \tilde{S}_j$. Because $\sigma^j_s(a) = \sigma^j_s(b)$, for $k \in F$, $b \notin X_{k,s}$. Hence $b \notin X_k$ for $k \in F$. Finally, we use $\rho^j_s(a) = \rho^j_s(b)$ to show $b \notin X_k$ for $k \in G$. Since $a \notin X_k \cup S_k$ and $k \in G$, $a \in Y_k \cap T_k$. Moreover, because $j < i$, $a \in P^j_s$, so already $a \in Y_{k,s} \cap T_{k,s}$. Now $\rho^j_s(a) = \rho^j_s(b)$ implies $b \in Y_{k,s} \cap T_{k,s}$, so $b \notin X_k \cup S_k$. Finally $b \notin C_s$, so $b \notin C_t$. All this implies that $b \in U^j - (X_0 \cup \cdots \cup X_j \cup R \cup C_t)$. □

Recall B is a *small subset* of A , denoted $B \subset_s A$, if $B \subseteq A$ and

$$(\forall U, V)[U \cap (A - B) \subseteq^* V \Rightarrow (U - A) \cup V \text{ r.e.}].$$

For completeness' sake we verify the following well-known facts.

Lemma 2.

(1) *If $B \subset_s A$, then each $Y \sqsubset A$ such that $Y \subseteq^* B$ must be recursive.*
(2) *If $B \subset_s A$, $B \subset_m A$ and the set $X \sqsubset A$ is non-recursive, then $X - B$ is non-r.e.*

Proof.
(1) Let $U = \omega, V = A - Y$. Then $U \cap (A - B) =^* A - B \subseteq^* V$, so $\overline{A} \cup V = \overline{Y}$ is r.e.
(2) If $X - B$ is r.e., then $Y := X \cap B \sqsubset A$, because $A - (X \cap B) = (A - X) \cup (X - B)$. So by (i), $X \cap B$ is recursive. Since X is non-recursive, $X - B$ is non-recursive, so we can choose an infinite recursive $R \subseteq X - B$. This contradicts $B \subset_m A$.

Lemma 3. *Let A be non-recursive. Then there is $C \subseteq A$ such that $(\forall X \sqsubset A$ non-recursive$)$ $(\exists T \subseteq X$ recursive$)$ $[T \cap C$ non-recursive$]$. A strictly increasing recursive sequence $b_0 < b_1 \subset \ldots$ such that $T = \{b_i : i \in \omega\}$ can be obtained uniformly in (an index for) X. We write $T = T_X$.*

Proof. Let $B \subset_{sm} A$. By an infinitary version of the proof of the Friedberg splitting theorem, in [So87], obtain a u.r.e. partition (P_k) of B such that

$$(1) \qquad (\forall W)(\forall k)[W - B \quad \text{non-recursive} \quad \Rightarrow W - P_k \quad \text{non-recursive}].$$

Let $C = \bigcup\{P_n : n \in K\}$. We claim that C is the desired set. First we show that for each k and each non-recursive $X \sqsubset A$, $X \cap P_k$ is infinite. By Lemma 2, $X - B$ is non-r.e. So, by (1), $X - P_k$ is non-r.e., so $X \cap P_k$ must be infinite.

Now define $T_X = \{b_0, b_1, \ldots\}$, where (b_k) is an effective strictly increasing sequence and $b_k \in X \cap P_k$. To do so, by induction over k, enumerate $X \cap P_k$ until a new element is found. If X is non-recursive, then T_X will be an infinite recursive subset of X. Moreover, $T_X \cap C \equiv_1 K$, so $T_X \cap C$ is non-recursive.

We now give a Lemma on how to appropximate Σ^0_3 sets.

Lemma 4. *If P is a Σ^0_3 set, then there is a u.r.e. sequence (Z_i) such that $Z_i \subseteq \{0, \cdots, i\}$ and*

a) $(\forall b \in P)$ *(a.e.i)* $[b \in Z_i]$
b) $(\exists^\infty i)$ $[Z_i \subseteq P]$

Remark. If $b \notin P$, then $b \notin Z_i$ infinitely many i, so $b \in P \Leftrightarrow$ (almost every i) $b \in Z_i$. Note that the right hand side is in Σ^0_3-form.

Proof. We first assume that P is a Σ^0_2 and show that there exist a sequence (Y_i) of strong indices for finite sets with the properties required in the lemma (this was first proved by Jockusch). For the general case, we will relativize this to $\emptyset'$.

If P is Σ^0_2, then there is an r.e. set C such that $P = \{(x)_0 : x \in \overline{C}\}$. Choose a recursive sequence of strong indices for sets $C_i \subseteq \{0, \cdots, i\}$ such that $C = \bigcup C_i$. Let $d(i) = min(C_{i+1} - C_i)$ if $C_{i+1} - C_i \neq \emptyset$ and $d(i) = i + 1$ else. Note that at most two arguments for the map d can yield the same value. Let Z_i be a strong index for

$$\{c < d(i) : c \notin C_i\}.$$

Then $a \notin C \Rightarrow a \in Z_i$ for almost every i and, if j is a non-deficiency state, i.e.

$$d(j) = \min\{d(i) : i \geq j\},$$

then $Z_j \subseteq \overline{C}$. Now let

$$Y_i = \{(x)_0 : x \in Z_i\}.$$

Then $Y_i \subseteq \{0, \cdots i\}$ (as this holds for Z_i). For (a), if $b \in P$, say $b = (c)_0$ for $c \in \overline{C}$, then $b \in Y_1$ whenever $d_i > c$, so for almost every i, $b \in Y_i$. For (b), note that $Y_i \subseteq P$ whenever $Z_i \subseteq \overline{C}$.

Now suppose P is Σ^0_3. By relativizing the Σ^0_2 case to $\emptyset'$, obtain a Δ^0_2-sequence of strong indices for finite sets $Y_i \subseteq \{0, \cdots, i\}$ such that (a) and (b) hold for (Y_i) in place of (Z_i). By the Limit lemma [So87], there is a recursive array of strong indices $(Y_{i,k})$ such that for each i and for almost every k, $Y_{i,k} = Y_i$. Let

$$Y^*_{\langle i,k\rangle} = \{0, \cdots, i\} \cap \bigcup_{t \geq k} Y_{i,t},$$

and let f be a 1-1 recursive function such that

$$rg(f) = \{\langle i, k\rangle : k = 0 \ \vee \ Y_{i,k} \neq Y_{i,k-1}\}.$$

Note that, for each i, there are only finitely many j such that $(f(j))_0 = i$. We claim that $Z_j = Y^*_{f(j)} \cap \{0, \cdots, j\}$ is the desired u.r.e. sequence. For (a), if $b \in P$, then for almost every i, $b \in Y_i$. Since $Y_i = Y^*_{i,k}$ for almost every k, by the above property of f, $b \in Z_j$ for almost every j.

For (b), if i is such that $Y_i \subseteq B$ and s is maximal such that $s = 0$ or $Y_{i,s-1} \neq Y_{i,s}$, then, for j such that $f(j) = \langle i, s\rangle$, $Z_j = Y_i \subseteq B$ (recall that $Y_i \subseteq \{0, \ldots, i\}$). Therefore $Z_j \subseteq B$ for infinitely many j.

We are now ready to carry out the inductive step in the proof of the Ideal Definability Lemma. We actually show the following: if $m \geq 2$ and I is an $m+3$-acceptable ideal of $\mathcal{B}(A)$, then there is a non-recursive $C \subseteq A$ and an $m+1$-acceptable ideal J of $\mathcal{B}(C)$ such that, for each $U \sqsubset A$,

(2) $$U \in I \Leftrightarrow (\exists R \subseteq A)(\forall S \subseteq A - R) \ [U \cap S \cap R \in J].$$

Let $P = \{e : U_e \in I\}$ (recall (U_e) is an enumeration of the halfs of splittings of A). By applying Lemma 4 relativized to $\emptyset^{(m)}$, we obtain a sequence of sets (Z_i) which are uniformly Σ^0_{m+1} such that $Z_i \subseteq \{0, \cdots i\}$ and

$$(a') \qquad U_e \in I \Leftrightarrow (a.e.\ i)(e \in Z_i]$$

$$(b') \qquad (\exists^\infty i)(\forall e \in Z_i)[U_e \in I].$$

Let $C \subseteq A$ be the set obtained by Lemma 3. Moreover let $\mathcal{B}(A)_{\leq i}$ be the Boolean algebra generated by $\{U_e : e \leq i\}$ (assume $B(A)_{\leq 0} = \{\emptyset, A\}$).

Claim. *There is a $\emptyset''$-sequence $(S_i)_{i\in\omega}$ of (indices for) recursive subsets of A such that the S_i are pairwise disjoint,* $(\forall R \subseteq A)(\exists i)[R \subseteq S_0 \cup \cdots \cup S_i]$ *and*

$$(\forall i)(\forall V \in \mathcal{B}(A)_{\leq i})[V \textit{ non-recursive } \Rightarrow V \cap S_i \cap C \textit{ non-recursive}].$$

Then we will define J essentially as the ideal on $\mathcal{B}(C)$ generated by the intersections $U_e \cap S_i \cap C$, where $e \in Z_i$. Let (R_i) be a $\emptyset''$-listing of (indices for) recursive sets for $\mathcal{R}(A)$.

Proof of the Claim.
Let $S_0 = \emptyset$ and, if $\hat{S}_i = S_0 \cup \cdots \cup S_i$,

$$S_{i+1} = (R_i - \hat{S}_i) \cup \bigcup \{T_{V-\hat{S}_i} : V \in \mathcal{B}(A)_{\leq i+1}\}$$

Then $R_i \subseteq S_0 \cup \cdots \cup S_{i+1}$. Moreover, if $V \in \mathcal{B}(A)_{\leq i+1}$ is non-recursive, then, by Lemma 3, $T_{V-\hat{S}_i} \cap C$ is non-recursive (where $T_{V-\hat{S}_i} \subseteq V$), so, since S_{i+1} recursively splits into $T_{V-\hat{S}_i}$ and $S_{i+1} - T_{V,\hat{S}_i}$, $V \cap S_{i+1} \cap C$ must be non-recursive.

Let J be the ideal of $\mathcal{B}(C)$ generated by $\mathcal{R}(C)$ and $\{U_e \cap S_i \cap C : e \in Z_i\}$. Since $m \geq 2$, the relation "$e \in Z_i$" is Σ^0_{m+1} and (S_i) is a $\emptyset''$ sequence of (indices for) recursive sets, J is an $m+1$-acceptable ideal. It remains to verify (2). Suppose $U = U_{\tilde{e}}$.

"$\Rightarrow$" If $U_{\tilde{e}} \in I$, choose i_0 such that $e \in Z_i$ for all $i > i_0$. We claim that $R = S_0 \cup \cdots \cup S_{i_0}$ is a witness for the right hand side in (2). If $S \subseteq A - R$, then $S \subseteq S_{i_0+1} \cup \cdots \cup S_j$ for some $j > i_0$. Now $U_{\tilde{e}} \cap S_i \cap C \in J$ for any $i > i_0$ so, $U_{\tilde{e}} \cap S \cap C \in J$.

"$\Leftarrow$" Suppose $U_{\tilde{e}} \notin I$. Given any $R \subseteq A$, choose k such that $R \subseteq S_0 \cup \cdots \cup S_k$. By (b'), there is an $i > k$ such that $Z_i \subseteq \{e : U_e \in I\}$, and also $U_{\tilde{e}} \in \mathcal{B}(A)_{\leq i}$. We show that S_i is a counterexample to the right hand side in (2), i.e. $U_{\tilde{e}} \cap S_i \cap C \notin J$. Let $V = U_{\tilde{e}} - \bigcup_{e \in Z_i} U_e$. Since $U_{\tilde{e}} \notin I$, V is a non-recursive element of $\mathcal{B}(A)_{\leq i}$. So $V \cap S_i \cap C$ is not recursive by the claim above. But, if $U_{\tilde{e}} \cap S_i \cap C \in J$, then, by the disjointness of the sets (S_j),

$$U_{\tilde{e}} \cap S_i \cap C \subseteq T \cup (\bigcup_{e \in Z_i} U_e \cap S_i \cap C)$$

for some recursive subset T of C. So $V \cap S_i \cap C$ is recursive as a split of C which is contained in the recursive subset T of C.

This concludes the proof of the Ideal Definability Lemma.

Added 4/97. In [N ta1] an alternative way is described to show uniform definability of 3-acceptable ideals of $\mathcal{B}(A)$. Choose $D \subset_{sm} A$. If I is a Σ^0_3–ideal of $\mathcal{B}(A)$, then there is C, $D \subseteq^* C \subseteq^* A$ such that $I = \{X \sqsubset A : X \cap C \subseteq^* D\}$. The proof of this is simpler than the proof of Lemma 1 above.

In a forthcoming paper [N ta3], the second author introduces a simplified coding configuration and uses it to show that the Π_6-theory of $\mathcal{E}^*$ as a lattice is undecidable.

References

[FV 59] Feferman, Vaught, *The first-order properties of algebraic systems*, Fund. Math. **47** (1959), 57-103.

[Fr 58] M. Friedberg, *Three theorems on recursive enumeration*, Journ. Symb. Logic **23** (1958), 309-316.

[Ha,So91] L. Harrington and R. Soare, *Post's program and incomplete recursively enumerable sets*, Proc. Natl. Acad. Sci. USA **88** (1992), 10242-10246.

[Ho,N ta] W. Hodges, A. Nies, *Interpreting infinite linear orders*, Logic Colloquium 95 Proceedings, Springer Verlag..

[La68] A. H. Lachlan, *The elementary theory of the lattice of recursively enumerable sets*, Duke Math. Journal (1968), 123-146.

[Lem 87] S. Lempp, *Hyperarithmetical index sets in recursion theory*, Trans. Amer. Math. Soc. **303** (1991), no. (2), 559-583.

[Ma 66] D.A. Martin, *Classes of recursively enumerable sets and degrees of unsolvability*, Z. Math. Logik Grundl. Math **12**, 295-310.

[N 94] A. Nies, *The last question on recursively enumerable many-one degrees*, Algebra i Logika **33 (5), transl. 7/1995** (1994), 550-563.

[N 95] A. Nies, *Relativizations of structures arising from computability theory*, In: S. B. Cooper e.a. (Eds.) Computability, enumerability, unsolvability, London Math. Soc. Lecture Notes Series **224**, Cambridge University Press, 219-232.

[N ta1] A. Nies, *Intervals of the lattice of computably enumerable sets and effective boolean algebras*, to appear in the Bulletin of the Lond. Math. Soc..

[N ta2] A. Nies, *Effectively dense Boolean algebras and their applications*, in preparation.

[N ta3] A. Nies, *Undecidable fragments of the theory of* $\mathcal{E}^*$, in preparation.

[N,S,Slta] A. Nies, R. Shore and T. Slaman, *Interpretability and definability in the recursively enumerable Turing -degrees* (to appear).

[Se 89] V. Selivanov, *Fine hierarchies of arithmetical sets and definable index sets*, Trudy Instituta Matematiki SO AN SSSR **12** (1989), 165-185.

[S 81] R. Shore, *The theory of the degrees below 0'*, J. London Math. Soc. **24** (1981), no. (2), 1-14.

[Sl,Wo86] T. A. Slaman and W. H. Woodin, *Definability in the Turing degrees*, Illinois Journal of Math. **30** (1986), no. (2), 320-334.

[So 74] R. I. Soare, *Automorphisms of the lattice of r.e. sets*, Ann. of Math. **100** (1974), no. (2), 60-120.

[So 87] R. I. Soare, *Recursively Enumerable Sets and Degrees: The Study of Computable Functions and Computably Generated Sets*, Perspectives in Mathematical Logic, Ω-Series, Springer-Verlag, Berlin, 1987.

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