THE JOURNAL OF SYMBOLIC LOGIC
Volume 57, Number 3, Sept. 1992

THE THEORY OF THE RECURSIVELY ENUMERABLE WEAK TRUTH-TABLE DEGREES IS UNDECIDABLE

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Abstract. We show that the partial order of Σ_3^0-sets under inclusion is elementarily definable with parameters in the semilattice of r.e. wtt-degrees. Using a result of E. Herrmann, we can deduce that this semilattice has an undecidable theory, thereby solving an open problem of P. Odifreddi.

The upper semilattice $\mathbf{R}_{\mathrm{wtt}}$ of r.e. weak truth-table (wtt) degrees has been investigated recently by several authors. Yet the question whether its elementary theory is undecidable, as posed first in [Od81], remained open. The first undecidability proof for the theory of the r.e. Turing degrees was announced in [Ha,Sh82]; a simpler one is presented in [ASp,Sh?]. The undecidability of the theory of the r.e. tt-degrees is proved in [Ht,S90]. However, the methods used in these proofs cannot be applied to establish the undecidability of Th($\mathbf{R}_{\mathrm{wtt}}$), since the r.e. wtt-degrees form a distributive semilattice. In this paper, we will show that the partial order $\mathscr{E}^3$ of Σ_3^0-sets under inclusion is elementarily definable with parameters (e.d.p.) in $\mathbf{R}_{\mathrm{wtt}}$, using distributivity in an essential way. The idea is to let Σ_3^0-sets correspond to certain ideals of $\mathbf{R}_{\mathrm{wtt}}$. These ideals can be represented by pairs of wtt-degrees, a fact which makes it possible to talk about them in the language of $\mathbf{R}_{\mathrm{wtt}}$. After this, the undecidability of Th($\mathbf{R}_{\mathrm{wtt}}$) can be deduced, using a general model theoretic theorem and E. Herrmann's result that all recursive Boolean pairs are, in a uniform way, e.d.p. in $\mathscr{E}^3$.

Outline of the paper. The undecidability proof is split up into its model theoretic, algebraic and recursion theoretic components. In the first section, we show that the elementary definability with parameters of $\mathscr{E}^3$ in $\mathbf{R}_{\mathrm{wtt}}$ implies the undecidability of Th($\mathbf{R}_{\mathrm{wtt}}$). The next section provides algebraic lemmas about upper semilattices. In §3, we prove two recursion theoretic results about the r.e. wtt-degrees. Finally, in §4, all this material is combined.

§0. Definitions, notation and conventions. An *upper semilattice with least element* (u.s.l.) is a structure $\mathbf{P} = (P; \leq, \vee, 0)$ such that 0 is the least element of the partial

Received March 5, 1991; revised June 24, 1991.

Part of this research was done while the authors visited the Mathematical Sciences Research Institute, in Berkeley, California. The first and the third author were supported by MSRI under NSF grant DMS-8505550, the third author in addition by NSF grant DMS-8912797. The second author was supported by the DAAD (Deutscher Akademischer Austauschdienst).

0022-4812/92/5703-0008/$02.10

order $(P; \leq)$, and, for a, $b \in P$, $a \vee b$ is the supremum of $\{a, b\}$. An *ideal* of a u.s.l. **P** is a nonempty subset which is closed downwards and under suprema. A *principal ideal* is an ideal of the form $[0, p]$ for some $p \in P$. The ideal generated by a subset A of P is denoted by $[A]_{\mathrm{id}}$. Note that

$$[A]_{\mathrm{id}} = \{p \in P: (\exists a_1, \ldots, a_n \in A)[p \leq \sup\{a_i: 1 \leq i \leq n\}]\}$$

(where $\sup(\emptyset) = 0$). An equation $a = b \wedge c$ in a u.s.l. is to be interpreted as "the infimum of $\{b, c\}$ exists and equals a". A u.s.l. **P** is called *distributive* if for $a, b, c \in P$ the following holds:

$$a \leq b \vee c \rightarrow \exists b_0 \exists c_0 (b_0 \leq b \wedge c_0 \leq c \wedge a = b_0 \vee c_0).$$

By induction, a similar property holds for suprema of finitely many elements of P.

For unexplained recursion theoretic notation, see [So87]. Given $n \geq 1$, $\mathscr{E}^n$ is the partial order of Σ_n^0-sets under inclusion. $\langle a, b\rangle$ is the ordered pair of the objects a, b. If a and b are natural numbers, then $\langle a, b\rangle$ is assumed to be the value of a fixed recursive pairing function applied to a and b. Moreover, in this case we require that $\langle a, b\rangle \geq \max\{a, b\}$. For a set X of natural numbers, $X^{[k]}$ denotes the set $\{u: \langle u, k\rangle \in X\}$.

As in [So87, p. 49], we make the following convention for the use $u(A; e, x, s)$ of a computation $\{e\}_s^A(x)$: if the computation is defined, then $u(A; e, x, s) \leq s$. For $e = \langle e_0, e_1\rangle$, we let

$$[e](x) = \{e_1\}(x), \qquad [e]_s(x) = \{e_1\}_s(x),$$

$$[e]^A(x) = \begin{cases} \{e_0\}^A(x) & \text{if } \{e_0\}^A(x)\downarrow \text{ and } u(A; e_0, x) \leq [e](x)\downarrow, \\ \text{undefined} & \text{otherwise}, \end{cases}$$

$$[e]_s^A(x) = \begin{cases} \{e_0\}_s^A(x) & \text{if } \{e_0\}_s^A(x)\downarrow \text{ and } u(A; e_0, x, s) \leq [e]_s(x)\downarrow, \\ \text{undefined} & \text{otherwise}. \end{cases}$$

Then A is *weak truth table* (*wtt*) *reducible* to B, written $A \leq_{\mathrm{wtt}} B$, if $A = [e]^B$ for some e. The wtt-degree of a set X of natural numbers is denoted by $\deg_{\mathrm{wtt}}(X)$. We use lower case boldface letters for wtt-degrees containing an r.e. set.

§1. The reduction scheme. The goal of this section is to show the following: if, for some $n \geq 1$, $\mathscr{E}^n$ is e.d.p. in a u.s.l. **P**, then the elementary theory of **P** is undecidable. First we give a precise definition of elementary definability with parameters, which is the restriction of a definition in [Bu,McK81] to languages L and L' with relation symbols only. Intuitively speaking, to define an L-structure A in an L'-structure D, we represent elements of A by m-tuples of elements of D, for some fixed $m \geq 1$. Then the relations in the L-structure A (including equality) give rise to corresponding relations between such m-tuples. We require that the set of m-tuples in D representing elements of A as well as these relations are definable in D with some fixed parameter list.

DEFINITION 1. (i) Let L and L' be languages of finite type with relation symbols only. A *scheme* s for interpreting L in L' consists of

natural numbers $n \geq 0$ and $m \geq 1$,
L'-formulas Un and Eq, and
L'-formulas φ_R for each relation symbol R in L,

such that (where $\tilde{x}^j = (x_{j\cdot m}, \ldots, x_{(j+1)\cdot m-1})$, $\bar{y} = (y_0, \ldots, y_{n-1})$, and all of these variables are assumed distinct)

$$\mathrm{Un} = \mathrm{Un}(\tilde{x}^0, \bar{y}), \qquad \mathrm{Eq} = \mathrm{Eq}(\tilde{x}^0, \tilde{x}^1, \bar{y})$$

and, for each q-ary relation symbol R of L,

$$\varphi_R = \varphi_R(\tilde{x}^0, \ldots, \tilde{x}^{q-1}, \bar{y}).$$

(ii) Let D be an L'-structure and $\bar{d}$ an n-tuple of elements of D. $(D, \bar{d})$ is said to *admit* the scheme s if the following holds (where $\tilde{c}$ always denotes an m-tuple of elements of D):

$$\{\langle \tilde{a}_0, \tilde{a}_1 \rangle : D \models \mathrm{Un}(\tilde{a}_0, \bar{d}) \wedge \mathrm{Un}(\tilde{a}_1, \bar{d}) \wedge \mathrm{Eq}(\tilde{a}_0, \tilde{a}_1, \bar{d})\} \tag{1.1}$$

is an equivalence relation, and, for each q-ary relation symbol R of L, the relation

$$\{\langle \tilde{a}_0, \ldots, \tilde{a}_{q-1} \rangle : D \models \mathrm{Un}(\tilde{a}_0, \bar{d}) \wedge \cdots \wedge \mathrm{Un}(\tilde{a}_{q-1}, \bar{d}) \wedge \varphi_R(\tilde{a}_0, \ldots, \tilde{a}_{q-1}, \bar{d})\} \tag{1.2}$$

is compatible with the equivalence relation (1.1), i.e., if, for all $i < q$, $D \models \mathrm{Eq}(\tilde{a}_i, \tilde{b}_i, \bar{d}) \wedge \mathrm{Un}(\tilde{a}_i, \bar{d}) \wedge \mathrm{Un}(\tilde{b}_i, \bar{d})$, then

$$D \models \varphi_R(\tilde{a}_0, \ldots, \tilde{a}_{q-1}, \bar{d}) \quad \text{iff} \quad D \models \varphi_R(\tilde{b}_0, \ldots, \tilde{b}_{q-1}, \bar{d}).$$

If $(D, \bar{d})$ admits s, then in a canonical way we can obtain a corresponding L-structure as follows: elements of the structure are the equivalence classes of the equivalence relation (1.1), and the relations are given by (1.2). We denote this structure by $D(s, \bar{d})$.

(iii) Let A be an L-structure. A is *elementarily definable with parameters* (e.d.p.) *in D via a scheme s*, if, for some n-tuple $\bar{d}$, $(D, \bar{d})$ admits s and $A \cong D(s, \bar{d})$. A class $\mathscr{K}$ of models for L is e.d.p. in a class $\mathscr{K}'$ of models for L' if there is a scheme s such that every model $A \in \mathscr{K}$ is e.d.p. in some model $D \in \mathscr{K}'$ via s.

Let L be as above. A class $\mathscr{K}$ of L-structures is *hereditarily undecidable* if every class of L-structures containing $\mathscr{K}$ has undecidable theory. In [Bu,McK81] there is a general theorem, which we now state in a simplified form.

THEOREM 1 (Burris and McKenzie). *Let L and L' be languages as above, and let $\mathscr{K}$ and $\mathscr{K}'$ be classes of models for L and L', respectively. Suppose that $\mathscr{K}$ is e.d.p. in $\mathscr{K}'$. If $\mathscr{K}$ is hereditarily undecidable, then so is $\mathscr{K}'$.*

We now provide an example of a hereditarily undecidable class which is very useful for undecidability proofs. A *Boolean pair* is a structure $(B; B_0, \leq)$ such that $(B; \leq)$ is the partial order of a Boolean algebra and B_0 is a subalgebra of this Boolean algebra. (Note that the Boolean operations of a Boolean algebra can be defined from the partial ordering.) A Boolean pair is *recursive* if, given some coding of B into the natural numbers, the Boolean operations and the set B_0 are recursive.

Burris and McKenzie [Bu,McK81] gave a certain class of Boolean pairs which is hereditarily undecidable. Herrmann [He84] observed that every Boolean pair in this class is recursive. Therefore, every class of Boolean pairs which contains the class of recursive Boolean pairs is hereditarily undecidable. Herrmann uses this fact, together with Theorem 1, to prove (ii) of the following theorem.

THEOREM 2 (E. Herrmann). *Let $n \geq 1$.*

(i) *There is a class C_n of Boolean pairs containing all recursive Boolean pairs such that C_n is e.d.p. in $\mathscr{E}^n$.*

(ii) *$\mathscr{E}^n$ (more precisely: the class $\{\mathscr{E}^n\}$) is hereditarily undecidable.*

PROOF. See [He84] and [He83].

In §4, we will show that $\mathscr{E}^3$ is e.d.p. in $\mathbf{R}_{\text{wtt}}$. By the next theorem, this suffices to establish the undecidability of $\text{Th}(\mathbf{R}_{\text{wtt}})$.

THEOREM 3. *Let $\mathbf{P} = (P; \leq, \vee, 0)$ be a u.s.l. such that, for some $n \geq 1$, $\mathscr{E}^n$ is e.d.p. in $\mathbf{P}$. Then the elementary theory of $\mathbf{P}$ is undecidable.*

PROOF. By (ii) of Theorem 2, $\mathscr{E}^n$ is hereditarily undecidable. Using Theorem 1, this property carries over to $\mathbf{P}$. Therefore $\text{Th}(\mathbf{P})$ is undecidable.

§2. Ideals and definability in upper semilattices. Throughout this section, let $\mathbf{P} = (P; \leq, \vee, 0)$ be a u.s.l. A subset A of P is called *independent* if, for every nonempty finite subset F of A and every $a \in A - F$, $a \not\leq \sup(F)$. Part (ii) of the next lemma contains a basic step of our proof that $\mathscr{E}^3$ is e.d.p. in $\mathbf{R}_{\text{wtt}}$: ideals generated by a subset of a countably infinite independent set can be viewed as subsets of ω.

LEMMA 1. *Let A be a countably infinite independent subset of P.*

(i) *If an ideal I of $\mathbf{P}$ is generated by a subset B of A, then $B = I \cap A$.*

(ii) *Let $\mathscr{C} = \{I: I$ is an ideal of $P \wedge I = [I \cap A]_{\text{id}}\}$ and let $(a_i)_{i \in \omega}$ be a sequence of pairwise different elements of P such that $A = \{a_i: i \in \omega\}$. The map $\phi: P(\omega) \to \mathscr{C}$ defined by $\phi(Z) = [\{a_i: i \in Z\}]_{\text{id}}$ is an isomorphism between $(P(\omega), \subseteq)$ and $(\mathscr{C}, \subseteq)$.*

PROOF. (i) Obviously $B \subseteq I \cap A$. For the converse inclusion, if $a \in I \cap A$, then $a \leq \sup(F)$ for some finite subset F of B. Since A is independent and $B \subseteq A$, we see that $a \in F$.

(ii) Clearly ϕ is surjective. Moreover, if $X \subseteq Y \subseteq \omega$, then $\phi(X) \subseteq \phi(Y)$. It remains to prove that $\phi(X) \subseteq \phi(Y)$ implies $X \subseteq Y$. Consider any subset Z of ω. By the definition of ϕ, $\{a_i: i \in Z\}$ generates $\phi(Z)$, whence by (i) $\{a_i: i \in Z\} = \phi(Z) \cap A$. Therefore $Z = \{i: a_i \in \phi(Z)\}$. Now assume that X and Y are subsets of ω such that $\phi(X) \subseteq \phi(Y)$. Then $X = \{i: a_i \in \phi(X)\} \subseteq \{i: a_i \in \phi(Y)\} = Y$. *q.e.d.*

In part (iii) of the following lemma, we give stronger conditions on the set A which imply that, for a certain subclass ID_A of the class $\mathscr{C}$ defined above, $(\text{ID}_A, \subseteq)$ is e.d.p. in $\mathbf{P}$. In §§3 and 4, we will investigate a subset A of $\mathbf{R}_{\text{wtt}}$ satisfying these conditions and show that, in the sense of Lemma 1(ii), ideals in ID_A correspond to Σ^0_3-sets.

We say that a subset A of P is *relatively definable* if it is first-order definable in the u.s.l. $([A]_{\text{id}}; \leq, \vee, 0)$.

LEMMA 2. *Let A be an independent subset of P, and let $S = \{\sup(F): F \subseteq A \wedge F$ is finite$\}$.*

(i) *For any ideal I of $\mathbf{P}$, I is generated by $I \cap A$ if and only if*

$$(\forall z \in I)(\exists s \in S)[s \in I \wedge z \leq s]. \tag{2.1}$$

(ii) *If A is relatively definable, then so is S.*

(iii) *Let* $I_0 = [A]_{\mathrm{id}}$ *and*

$$\mathrm{ID}_A = \{I: I \text{ is an ideal of } \mathbf{P} \wedge I = [I \cap A]_{\mathrm{id}} \wedge (\exists u, v \in P)(I = [0,u] \cap [0,v])\}.$$

Suppose that $I_0 \in \mathrm{ID}_A$ *and that* A *is relatively definable. Then* $(\mathrm{ID}_A, \subseteq)$ *is e.d.p. in* $(P, \leq)$.

PROOF. (i) For one direction, suppose that I is generated by $I \cap A$, and let $z \in I$ be arbitrary. There exist a finite set $F \subseteq I \cap A$ such that $z \leq s = \sup(F)$. Then $s \in I$. This shows (2.1). For the other direction, suppose that (2.1) holds and let $z \in I$. Choose $s \in S$ such that $s \in I$ and $z \leq s$. By definition, $s = \sup(F)$ for some finite subset of A. Then $F \subseteq I \cap A$. Hence I is generated by $I \cap A$.

(ii) We show that, for each $x \in [S]_{\mathrm{id}} = [A]_{\mathrm{id}}$, $x \in S$ if and only if

$$(\forall y < x)(\exists a \in A)[a \leq x \wedge a \not\leq y]. \tag{2.2}$$

This will imply (ii).

For one direction, suppose that $x \in S$. Then $x = \sup(F)$ for some finite subset F of A. Given $y < x$, there exists $a \in F$ such that $a \not\leq y$. This shows (2.2).

For the other direction, suppose that $x \notin S$. Since $x \in [S]_{\mathrm{id}}$, $x \leq \sup(F)$ for some finite subset F of A. Let $y = \sup(\{a \in F: a \leq x\})$. Then $y < x$. Now, if $a \in A$ and $a \leq x$, then $a \in F$ by independence of A. Hence $a \leq y$. This shows that (2.2) fails.

(iii) First we consider ideals I of $\mathbf{P}$ which can be represented by a pair v, w of elements of P in the sense that $I = [0,v] \cap [0,w]$. The formula

$$\varphi_{\leq}(x,y,x',y') \equiv (\forall z)[(z \leq x \wedge z \leq y) \rightarrow (z \leq x' \wedge z \leq y')]$$

describes inclusion of represented ideals: For all $v, w, v', w' \in P$

$$\mathbf{P} \models \varphi_{\leq}(v,w,v',w') \text{ iff } [0,v] \cap [0,w] \subseteq [0,v'] \cap [0,w']. \tag{2.3}$$

Then the formula

$$\mathrm{Eq}(x,y,x',y') \equiv \varphi_{\leq}(x,y,x',y') \wedge \varphi_{\leq}(x',y',x,y)$$

describes equality of represented ideals.

Finally we give a formula Un defining the set

$$E = \{\langle v,w \rangle: [0,v] \cap [0,w] \in \mathrm{ID}_A\}$$

in $\mathbf{P}$ with parameters. Choose elements r_0 and r_1 of P such that $I_0 = [0,r_0] \cap [0,r_1]$. Since A is relatively definable, (ii) shows that the set S is first-order definable with parameters in $\mathbf{P}$ by a formula $\varphi_S(x, p_0, p_1)$ (substituting the parameters r_0 and r_1 for the variables p_0 and p_1). To obtain a formula with parameters defining E in $\mathbf{P}$, we transcribe (2.1), replacing ideals by pairs of variables. The resulting formula is

$$\mathrm{Un}(x,y,p_0,p_1) \equiv (\forall z)[(z \leq x \wedge z \leq y) \rightarrow (\exists s)[\varphi_S(s,p_0,p_1) \wedge s \leq x \wedge s \leq y \wedge z \leq s]].$$

So, by (i),

$$\langle v,w \rangle \in E \quad \text{iff} \quad \mathbf{P} \models \mathrm{Un}(v,w,r_0,r_1). \tag{2.4}$$

It is easy to see that the formulas Un, Eq and $\varphi_{\leq}$ form a scheme s in the sense of Definition 1 (with $m = n = 2$), which is admitted by $(\mathbf{P}, \langle r_0, r_1 \rangle)$. Since, by (2.3) and (2.4), $(\mathrm{ID}_A, \subseteq) \cong \mathbf{P}(s, \langle r_0, r_1 \rangle)$, this implies that $(\mathrm{ID}_A, \subseteq)$ is e.d.p. in $\mathbf{P}$. *q.e.d.*

We now give a sufficient condition on a subset A of a *distributive* u.s.l. which implies that A is independent and relatively definable. For an element p of P we write $\mathrm{ncl}(p)$ if $p \neq 0$ and the u.s.l. $[0, p]$ is nowhere complemented, i.e. there are no $r, s \in P - \{0\}$ such that $r \vee s = p$ and $r \wedge s = 0$.

LEMMA 3. *Let* $\mathbf{P} = (P; \leq, \vee, 0)$ *be a distributive u.s.l.*

(i) *Suppose that* A *is a nonempty subset of* $P - \{0\}$ *such that*

$$a, b \in A \wedge a \neq b \rightarrow a \wedge b = 0. \tag{2.5}$$

If $a, a_1, \ldots, a_n$ *are pairwise distinct elements of* A, *then* $a \wedge \sup\{a_i\colon 1 \leq i \leq n\} = 0$. *In particular,* A *is independent.*

(ii) *Let* A *be a nonempty subset of* P *satisfying* (2.5) *and* $a \in A \rightarrow \mathrm{ncl}(a)$. *Then* A *is relatively definable.*

PROOF. (i) Assume for a contradiction that there is an element u of P such that $u \neq 0$, $u \leq a$ and $u \leq \sup\{a_i\colon 1 \leq i \leq n\}$. By distributivity, $u = \sup\{b_i\colon 1 \leq i \leq n\}$ for some elements b_i of P, $b_i \leq a_i$ $(1 \leq i \leq n)$. Hence there is an index k, $1 \leq k \leq n$, such that $b_k \neq 0$. Since $b_k \leq a$, this contradicts $a \wedge a_k = 0$.

(ii) Let $\varphi_{\mathrm{ncl}}(x)$ be a formula defining the property $\mathrm{ncl}(p)$ of an element p of P in the language of upper semilattices. We claim that the formula

$$\varphi_{\mathrm{ncl}}(x) \wedge (\forall y)[x \leq y \wedge \varphi_{\mathrm{ncl}}(y) \rightarrow x = y] \tag{2.6}$$

defines A in the u.s.l. $([A]_{\mathrm{id}}; \leq, \vee, 0)$.

First we show that for every $p \in [A]_{\mathrm{id}}$ the following holds:

$$\mathrm{ncl}(p) \rightarrow (\exists a)[a \in A \wedge p \leq a]. \tag{2.7}$$

Choose $a_1, \ldots, a_n \in A$ pairwise different such that $p \leq \sup\{a_i\colon 1 \leq i \leq n\}$. If $n = 1$, then we are finished. So let $n > 1$. By distributivity of $\mathbf{P}$, there are b_i $(1 \leq i \leq n)$ such that $b_i \leq a_i$ and $p = \sup\{b_i\colon 1 \leq i \leq n\}$. Since $p \neq 0$, some component b_i of p is nonzero, say b_1. By (i) and (2.5), $b_1 \wedge \sup\{b_i\colon 2 \leq i \leq n\} = 0$. Since $\mathrm{ncl}(p)$ holds, this implies $\sup\{b_i\colon 2 \leq i \leq n\} = 0$ and therefore $p \leq a_1$. By (2.7), an element of $[A]_{\mathrm{id}}$ satisfying the formula (2.6) must be in A. On the other hand, if $a \in A$ does not satisfy this formula, then there is $p \in [A]_{\mathrm{id}}$ such that $\mathrm{ncl}(p)$ and $a < p$. By (2.7), $p \leq b$ for some $b \in A$, contradicting (2.5). *q.e.d.*

From (iii) of Lemma 2 and (ii) of Lemma 3, we obtain the following theorem.

THEOREM 4. *Let* $\mathbf{P} = (P; \leq, \vee, 0)$ *be a distributive u.s.l., and let* A *be a nonempty subset of* $P - \{0\}$ *which satisfies* $a, b \in A \wedge a \neq b \rightarrow a \wedge b = 0$ *and* $a \in A \rightarrow \mathrm{ncl}(a)$. *Let*

$$\mathrm{ID}_A = \{I\colon I = [I \cap A]_{\mathrm{id}} \wedge (\exists u, v \in P)(I = [0, u] \cap [0, v])\}.$$

If $[A]_{\mathrm{id}} \in \mathrm{ID}_A$, *then* $(\mathrm{ID}_A, \subseteq)$ *is e.d.p. in* $\mathbf{P}$.

§3. Preliminary results about r.e. wtt-degrees. In this section, we provide results about the r.e. wtt-degrees which will be used later (1) to construct a set $\mathbf{A}$ of r.e. wtt-degrees satisfying the hypotheses of Theorem 4 (where $\mathbf{P} = \mathbf{R}_{\mathrm{wtt}}$), and (2) to guarantee in addition that $\mathscr{E}^3 \cong (\mathrm{ID}_A, \subseteq)$.

For (1), we cite a lemma of A. Lachlan. Further on, we verify that a theorem by Ambos-Spies and Soare about r.e. Turing degrees carries over to the r.e. wtt-degrees. For (2), we characterize the ideals of $\mathbf{R}_{\mathrm{wtt}}$ which can be represented as the intersection of two principal ideals.

LEMMA 5 (Lachlan). *The u.s.l.* $\mathbf{R}_{\mathrm{wtt}}$ *is distributive.*

PROOF. See [La72].

We review some definitions. A pair A, B of r.e. sets is called a *minimal pair* if A and B are nonrecursive and every r.e. set X which is T-reducible to A and to B is recursive. A nonrecursive r.e. set C is *nonbounding* if there is no minimal pair A, B, such that $A, B \leq_{\mathrm{T}} C$. These notions make sense as well if we replace Turing reducibility by wtt-reducibility. To emphasize the difference we will write *T-minimal pairs*, *wtt-minimal pairs*, etc.

An r.e. Turing degree is called *contiguous* if it contains only one r.e. wtt-degree. By [Ld,Sa75], for every nonrecursive r.e. set B there is a nonrecursive r.e. set A such that $A \leq_{\mathrm{wtt}} B$ and $\deg_{\mathrm{T}}(A)$ is contiguous. From this we can infer the following lemma.

LEMMA 6. *If an r.e. set C is T-nonbounding, then it is wtt-nonbounding.*

PROOF. Assume for contradiction that C is T-nonbounding and there are r.e. sets U and V which are wtt-reducible to C and form a wtt-minimal pair. Choose nonrecursive r.e. sets $U_1 \leq_{\mathrm{wtt}} U$ and $V_1 \leq_{\mathrm{wtt}} V$ whose T-degrees are contiguous. Then U_1 and V_1 form a T-minimal pair bounded by C, since for each r.e. set X

$$\begin{aligned} X \leq_{\mathrm{T}} U_1, V_1 &\Rightarrow X \oplus U_1 \equiv_{\mathrm{T}} U_1 \text{ and } X \oplus V_1 \equiv_{\mathrm{T}} V_1 \\ &\Rightarrow X \oplus U_1 \equiv_{\mathrm{wtt}} U_1 \text{ and } X \oplus V_1 \equiv_{\mathrm{wtt}} V_1 \\ &\Rightarrow X \leq_{\mathrm{wtt}} U_1, V_1 \\ &\Rightarrow X \text{ is recursive.} \end{aligned}$$

q.e.d.

THEOREM 5 (Ambos-Spies and Soare). *There is an u.r.e. sequence* (A_i) $(i \in \omega)$ *of r.e. sets with the following properties*:

(i) $\deg_{\mathrm{wtt}}(A_i)$ *is wtt-nonbounding for every i.*

(ii) $i \neq j \rightarrow A_i$ *and* A_j *form a wtt-minimal pair.*

PROOF. By [ASp,So89], there is a sequence of r.e. sets satisfying (i) and (ii) with wtt-reducibility replaced by Turing reducibility. By the preceding lemma, (i) holds for this sequence. Since every T-minimal pair is a wtt-minimal pair, (ii) holds as well. By the construction in [ASp,So89], the sequence is u.r.e. *q.e.d.*

LEMMA 7 (Cohen). *The relation* $\{\langle k,l\rangle : W_k \leq_{\mathrm{wtt}} W_l\}$ *is* Σ^0_3.

PROOF.

$$W_k \leq_{\mathrm{wtt}} W_l \Leftrightarrow (\exists e)(\forall x)(\forall t)(\exists s \geq t)[[e]^{W_{l,s}}_S(x)\downarrow = W_{k,s}(x)].$$

Since the matrix of this expression is recursive, the whole expression is in Σ^0_3-form.

THEOREM 6. *Let* $\mathbf{I}$ *be an ideal of* $\mathbf{R}_{\mathrm{wtt}}$. *Then the following are equivalent*:

(i) *There are* $\mathbf{a}, \mathbf{b} \in \mathbf{R}_{\mathrm{wtt}}$ *such that* $\mathbf{I} = [\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$.

(ii) *The set* $\{e : \deg_{\mathrm{wtt}}(W_e) \in \mathbf{I}\}$ *is* Σ^0_3.

(iii) *There is an r.e. set U representing* $\mathbf{I}$ *in the sense that*

$$\mathbf{I} = \{\deg_{\mathrm{wtt}}(W_e) : e \in U\}.$$

An ideal of $\mathbf{R}_{\mathrm{wtt}}$ satisfying one of these equivalent conditions will be called a Σ^0_3*-ideal.*

PROOF. For the implication (i) $\rightarrow$ (ii), choose r.e. sets $W_n \in \mathbf{a}$ and $W_m \in \mathbf{b}$. Then

$$\{e : \deg_{\mathrm{wtt}}(W_e) \in \mathbf{I}\} = \{e : W_e \leq_{\mathrm{wtt}} W_n \wedge W_e \leq_{\mathrm{wtt}} W_m\}.$$

By Lemma 7, this set is Σ^0_3.

The implication (ii) $\to$ (iii) is a special case of a theorem by Yates (see [So87, p. 253]).

It remains to show the implication (iii) $\to$ (i). Since U is nonempty, there is a recursive function g such that $U = \mathrm{rg}(g)$. Let $X_n = W_{g(n)}$ and $X_{n,s} = W_{g(n),s}$. It suffices to construct r.e. sets A and B satisfying for every n the infinitary positive requirements

$$P_n^A\colon A^{[n]} =^* X_n, \qquad P_n^B\colon B^{[n]} =^* X_n,$$

and for every $k = \langle e, i\rangle$ the negative requirement

$$N_k\colon f = [e]^A = [i]^B,\ f \text{ total} \Rightarrow f \leq_{\mathrm{wtt}} X_0 \oplus \cdots \oplus X_{k-1}.$$

The negative requirements are satisfied by the usual minimal pair strategy (see [So87, Theorem IX.1.2]). We associate with every $k = \langle e, i\rangle$ a function $l(k, s)$ measuring the length of agreement between the computations $[e]^A$ and $[i]^B$ at stage s. Given $\{A_t\colon t \leq s\}$ and $\{B_t\colon t \leq s\}$, let

$$l(k,s) = \max\{x\colon (\forall y < x)([e]_S^{A_s}(y) = [i]_S^{B_s}(y)\downarrow)\}.$$

Based on this definition of l, as in [So87] we define k-expansionary stages and the restraint function $r(k,s)$. Stage s is *0-expansionary* if $(\forall t < s)[l(0,t) < l(0,s)]$ and

$$r(0,s) = \begin{cases} 0 & \text{if } s \text{ is 0-expansionary,} \\ \text{the greatest 0-expansionary stage } t < s & \text{otherwise.} \end{cases}$$

For $k > 0$, stage s is *k-expansionary* if

$$(\forall t < s)(r(k-1,t) = r(k-1,s) \to l(k,t) < l(k,s)),$$

and $r(k,s)$ is the maximum of

(a) $\quad r(k-1, s)$,

(b) $\quad$ those $t < s$ such that $r(k-1,t) < r(k-1,s)$,

(c) $\quad$ those $t < s$ such that $r(k-1,t) = r(k-1,s)$ and t is k-expansionary, if s is not k-expansionary.

Note that $[e]^A = [i]^B$ implies that there are infinitely many k-expansionary stages.

Construction of A and B. Stage 0. Let $A_0 = B_0 = \varnothing$.

Stage $s + 1$. If there is $x < s$, $x = \langle y, k\rangle$ such that $y \in X_{k,s}$, $x \notin A_s \cap B_s$ and $x \geq r(k,s)$, then choose a minimal x with these properties. Enumerate x in A if $x \notin A_s$, and in B otherwise. (For $x = \langle y, k\rangle$, we call this enumeration of x an *action for* P_n^A or P_n^B, respectively).

For a proof that the constructed sets have the required properties, first, exactly as in [So87], one can show that for every n

$$r(n) = \liminf_s r(n,s) \text{ exists.}$$

Next, by construction

$$\langle y, n\rangle \geq r(n) \to (y \in X_n \leftrightarrow y \in A^{[n]} \leftrightarrow y \in B^{[n]}). \tag{3.1}$$

Therefore the requirements P_n^A and P_n^B are met.

Finally, to show that the requirements N_k are met, fix $k = \langle e, i\rangle$ such that $f = [e]^A = [i]^B$ is total. We give a wtt-reduction of f to $X_0 \oplus \cdots \oplus X_{k-1}$.

Since the recursive bounds $[e]$ and $[i]$ on the use functions for $[e]^A$ and $[i]^B$ are total, we can define the recursive function

$$m(x) = \max\{[e](x), [i](x)\}.$$

Choose s_0 such that $(\forall s \geq s_0)(\forall i \leq k)[r(i,s) \geq r(i)]$.

Our reduction is as follows: given input x, compute the least $t \geq s_0$ such that $r(k,t) = r(k)$, t is k-expansionary, $l(k,t) > x$ and

$$(\forall i < k)(\forall y)[(\langle y, i\rangle \in [r(i), m(x)) \wedge y \in X_i) \rightarrow (y \in X_{i,t} \wedge \langle y, i\rangle \in A_t \wedge \langle y, i\rangle \in B_t)]. \tag{3.2}$$

(Here $X_0 \oplus \cdots \oplus X_{k-1}$ serves as an oracle, and the queries are bounded by a recursive function in x. By (3.1), t exists.) We claim that

$$f(x) = [e]_t^{A_t}(x).$$

To show that the last equation holds, let $t = t_0 < t_1 < \cdots$ be the k-expansionary stages $\geq t$. The equation will follow, if

(3.3) between two consecutive k-expansionary stages t_n and t_{n+1} one of the computations $[e]_{t_n}^{A_{t_n}}(x)$ or $[i]_{t_n}^{B_{t_n}}(x)$ is not destroyed by changing answers of the oracle to queries;

in this case $f(x) = \lim_n [e]_{t_n}^{A_{t_n}}(x) = [e]_t^{A_t}(x)$, since at k-expansionary stages $\geq t$ both computations must agree.

By the following two facts (3.3) holds:

(1) Because of (3.2), no action for a requirement P_i^A or P_i^B $(i < k)$ can destroy a computation after t.

(2) Because of (c) in the definition of $r(k,s)$, the only possible stages $\geq t$ where one computation can be destroyed are the stages $t_n + 1$. If at such a stage $[e]_{t_n}^{A_{t_n}}(x)$ is destroyed, say, then the other side is protected until stage t_{n+1} (where both sides must agree again). *q.e.d.*

§4. Undecidability of Th($\mathbf{R}_{\mathrm{wtt}}$).

THEOREM 7. *The partial order $\mathscr{E}^3$ of Σ_3^0-sets under inclusion is e.d.p. in* $\mathbf{R}_{\mathrm{wtt}}$.

PROOF. Let $(A_i)_{(i \in \omega)}$ be a u.r.e. sequence as in Theorem 5, $\mathbf{a}_i = \deg_{\mathrm{wtt}}(A_i)$ and $\mathbf{A} = \{\mathbf{a}_i : i \in \omega\}$. Let

$$\mathrm{ID}_{\mathbf{A}} = \{\mathbf{I} : \mathbf{I} = [\mathbf{I} \cap \mathbf{A}]_{\mathrm{id}} \wedge (\exists \mathbf{u}, \mathbf{v} \in \mathbf{R}_{\mathrm{wtt}})(\mathbf{I} = [\mathbf{0}, \mathbf{u}] \cap [\mathbf{0}, \mathbf{v}])\}.$$

By Theorem 6, $\mathrm{ID}_{\mathbf{A}}$ is the set of Σ_3^0-ideals generated by their intersections with $\mathbf{A}$.

Note that, by distributivity of $\mathbf{R}_{\mathrm{wtt}}$ and by (i) of Lemma 3, the set $\mathbf{A}$ is independent. Therefore, as in Lemma 1(ii), we can define an isomorphism ϕ between $(\mathscr{P}(\omega), \subseteq)$ and the set of ideals of $\mathbf{R}_{\mathrm{wtt}}$ generated by their intersection with $\mathbf{A}$: for any subset Z of ω, $\phi(Z) = [\{\mathbf{a}_i : i \in Z\}]_{\mathrm{id}}$. We claim that the restriction of the map ϕ to the Σ_3^0-sets yields an isomorphism $\mathscr{E}^3 \rightarrow (\mathrm{ID}_{\mathbf{A}}, \subseteq)$. It is sufficient to show that

$$\text{every } \Sigma_3^0\text{-set is mapped to a } \Sigma_3^0\text{-ideal} \tag{4.1}$$

and

$$\text{the preimage of every } \Sigma_3^0\text{-ideal is a } \Sigma_3^0\text{-set.} \tag{4.2}$$

Proof of (4.1). Let the variable σ range over tuples of natural numbers, and let λ denote the empty tuple. Since the sequence (A_i) is u.r.e., we can choose a recursive function g such that $W_{g(\lambda)} = \varnothing$ and, for $n = |\sigma| \geq 1$, $W_{g(\sigma)} = A_{\sigma(0)} \oplus \cdots \oplus A_{\sigma(n-1)}$.

Now let Z be any Σ^0_3-set and let

$$U = \{e: (\exists\sigma)[W_e \leq_{\mathrm{wtt}} W_{g(\sigma)} \wedge (\forall y < |\sigma|)[\sigma(y) \in Z]]\}.$$

Then $e \in U \Leftrightarrow \deg_{\mathrm{wtt}}(W_e) \in \phi(Z)$. By Lemma 7, U is a Σ^0_3-set. Therefore, $\phi(Z)$ is a Σ^0_3-ideal.

Proof of (4.2). Fix a recursive function g such that $A_i = W_{g(i)}$. Given $\mathbf{I} \in \mathrm{ID}_{\mathbf{A}}$, the set $V = \{e: \deg_{\mathrm{wtt}}(W_e) \in \mathbf{I}\}$ is Σ^0_3, whence $Z = g^{-1}(V)$ is Σ^0_3 as well. Since $\mathbf{I}$ is generated by $\mathbf{I} \cap \mathbf{A}$, we have $\varnothing(Z) = \mathbf{I}$.

Now we verify that the hypotheses of Theorem 4 with $\mathbf{P} = \mathbf{R}_{\mathrm{wtt}}$ are satisfied. This will show that $(\mathrm{ID}_{\mathbf{A}}, \subseteq)$ and therefore $\mathscr{E}^3$ is e.d.p. in $\mathbf{R}_{\mathrm{wtt}}$.

If $\mathbf{a}, \mathbf{b} \in \mathbf{A}$ and $\mathbf{a} \neq \mathbf{b}$, then $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$ by (ii) of Theorem 5. Condition (i) of the same theorem implies that $\mathrm{ncl}(\mathbf{a})$ holds for every $\mathbf{a} \in \mathbf{A}$. Since ω is a Σ^0_3-set and $[\mathbf{A}]_{\mathrm{id}} = \phi(\omega)$, (4.1) shows that $[\mathbf{A}]_{\mathrm{id}}$ is a Σ^0_3-ideal, whence $[\mathbf{A}]_{\mathrm{id}} \in \mathrm{ID}_{\mathbf{A}}$. Therefore all the hypotheses of Theorem 4 are satisfied. *q.e.d.*

COROLLARY. *The elementary theory of the r.e. weak truth-table degrees is undecidable.*

PROOF. By Theorems 3 and 7.

Note. Our method can be applied to other reducibility notions. For instance, in [ASp,N?] the first two authors have proved in an analogous way that the theory of the polynomial m-degrees of recursive sets is undecidable.

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