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Note

Recent examples in the theory of partition graphs

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Abstract

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A partition graph is an intersection graph for a collection of subsets of a universal set S with the property that every maximal independent set of vertices corresponds to a partition of S. Two questions which arose in the study of partition graphs are answered by recently discovered examples. An enumeration of the partition graphs on ten or fewer vertices is provided.

1. The resolution of two questions in partition graph theory

A graph G is a *general partition graph* [2] if there is some set S and an assignment of subsets $S_v \subset S$ to the vertices $v \in V(G)$ such that:

(i) $uv \in E(G)$ if and only if $S_u \cap S_v \neq \emptyset$;

(ii) $S = \bigcup_{v \in V(G)} S_v$; and

(iii) every maximal independent set of vertices M partitions S into a disjoint collection $\{S_m : m \in M\}$.

If, furthermore,

(iv) $v \neq v \Rightarrow S_u \neq S_v$,

then G is a *partition graph* [1, 2].

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It has been shown in [2] that $S_u = S_v$ in a general partition graph precisely when the vertices u and v have the same closed neighborhoods, so that $N[u] = N[v]$. Moreover if a graph G has two vertices u and v for which $N[u] = N[v]$, then G is a general partition graph if and only if $G - u$ is a general partition graph. Thus to ascertain if a graph is a general partition graph it suffices to remove, successively, all vertices which have the same closed neighborhood as another vertex, and then to check if the reduced graph (with distinct closed neighborhoods) is a partition graph.

In this note we answer two questions which arose in the study of partition graphs. The questions pertain to the conditions below on a graph G. In Condition I a *clique cover of* G is a collection of cliques for which every vertex and edge of G is a member of some clique in the collection.

Triangle Condition T. For every maximal independent set M of vertices in G and every edge uv in $G - M$, there is a vertex $m \in M$ such that uvm is a triangle in G.

Incidence Condition I. There is a clique cover Γ of G with the property that every maximal independent set has a vertex from each clique in Γ.

Condition T is known [1, 2] to be necessary for a graph to be a general partition graph. Condition I is both necessary and sufficient [2].

Condition T has been useful screening criterion to check if a given graph might be a partition graph. Indeed, since no contrary examples were known, it had been asked ([1, 2, 4]): Is Condition T also a sufficient condition?

A computer search of all graphs through ten vertices has shown that Condition T is not sufficient. The smallest example occurs on nine vertices, and is the graph G_T shown in Fig. 1.

There are five maximal independent sets of vertices, $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$, $\{v_1, v_4, v_7\}$, $\{v_2, v_5, v_8\}$ and $\{v_3, v_6, v_9\}$, and it is straightforward to verify that Condition T holds for G_T.

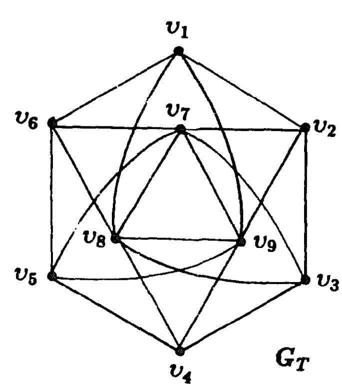

Fig. 1. A graph satisfying Condition T which is not a general partition graph.

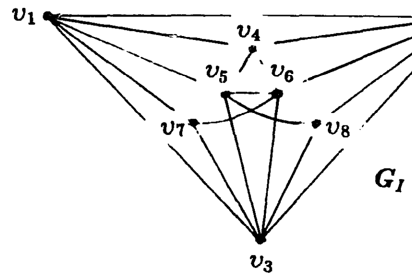

Fig. 2. A partition graph with two minimal clique covers, only one of which satisfies Condition I.

To see that G_T is not a partition graph we will show that Condition I is not met. Consider edge v_8v_9, which belongs to the three cliques $C_1 = v_1v_8v_9$, $C_2 = v_4v_8v_9$, $C_3 = v_7v_8v_9$ and no others. Thus every clique cover must include at least one of these three cliques. However, the maximal independent set $\{v_1, v_3, v_5\}$ has no vertex in either C_2 or C_3, and the maximal independent set $\{v_2, v_4, v_6\}$ has no vertex in C_1. Thus Condition I does not hold and so G_T is not a general partition graph.

G_T is the only example among the graphs of nine or fewer vertices for which Condition T is satisfied and yet the graph is not a general partition graph. There are nine additional examples on graphs with ten vertices. Two of these have a pair of vertices with identical closed neighborhoods, and both graphs reduce to G_T when one vertex of the pair is removed.

A clique cover is *minimal* if no proper subcollection of its cliques is also a clique cover. If a clique cover which satisfies Condition I exists, then it can be taken to be minimal. This raises the following question ([2, 4]): It is possible for a graph to have two different minimal clique covers, one which satisfies Condition I and one which does not!

The answer is now known to be no, and the graph G_I shown in Fig. 2 provides an example. Other examples can be found on eight vertices, but no examples occur on fewer vertices.

It is easy to check that

$$\Gamma_1 = \{v_1v_2v_3,\ v_1v_3v_7,\ v_1v_4v_5,\ v_2v_3v_8,\ v_2v_4v_6,\ v_3v_5v_8,\ v_3v_6v_7,\ v_4v_5v_6\},$$

is a minimal clique cover. However the maximal independent set $\{v_4, v_7, v_8\}$ has no vertex in the clique $v_1v_2v_3 \in \Gamma_1$. Nevertheless Condition I is satisfied for the minimal clique cover

$$\Gamma_2 = \{v_1v_2v_4,\ v_1v_3v_7,\ v_1v_4v_5,\ v_2v_3v_8,\ v_2v_4v_6,\ v_3v_5v_8,\ v_3v_6v_7,\ v_4v_5v_6\},$$

as can be readily verified. There are four maximal independent sets, $\{v_4, v_7, v_8\}$, $\{v_1, v_6, v_8\}$, $\{v_2, v_5, v_7\}$, and $\{v_3, v_4\}$, to be considered.

Table 1
Enumeration of connected partion graphs

Order p	1	2	3	4	5	6	7	8	9	10
Number of Partition Graphs	1	0	1	2	5	14	47	186	894	5249
Number of General Partition Graphs	1	1	2	5	13	41	145	604	2938	16947

2. Enumeration of partition graphs on p vertices, $p \leq 10$

It would be of interest to know, or at least have estimates or bounds, on the number of partition graphs of order p. A search through the adjacency matrices on all graphs on $p = 10$ or fewer vertices has provided Table 1. It should be observed that a graph is a (general) partition graph if and only if the same is true of each of its components, so the table is for connected graphs.

Depictions of the partition graphs through $p = 6$ vertices are found in [3; Table 1].

References

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