Finding Paths in Graphs

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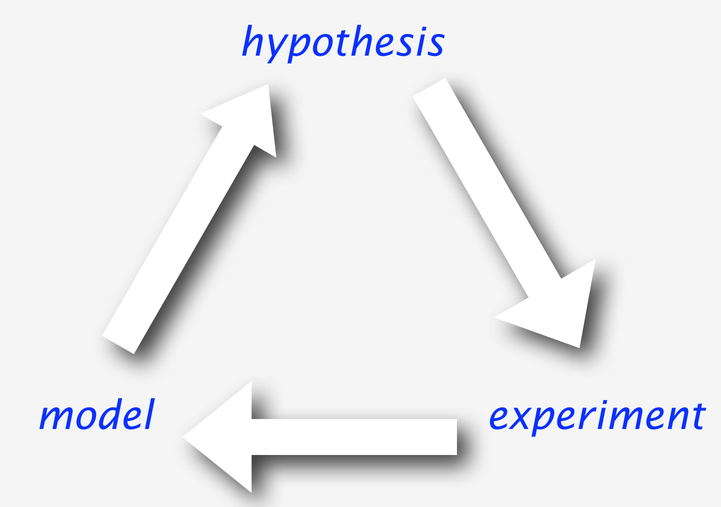
Subtext: the scientific method

is necessary in algorithm design and implementation

Scientific method
- create a model describing natural world
- use model to develop hypotheses
- run experiments to validate hypotheses
- refine model and repeat



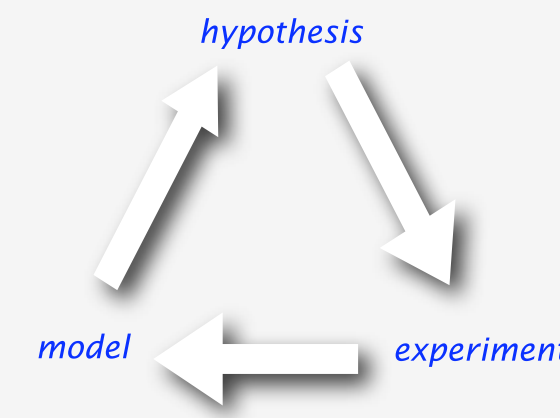
Algorithm designer who does not run experiments risks becoming lost in abstraction

Software developer who ignores resource consumption risks catastrophic consequences

Isolated theory or experiment can be of value when clearly identified

Warmup: random number generation

Problem: write a program to generate random numbers



model: classical probability and statistics

hypothesis: frequency values should be uniform

weak experiment:

- generate random numbers
- check for uniform frequencies

```
int k = 0;
while ( true )
  System.out.print(k++ % V);
```

0 1 2 3 4 5 6 7 8 9 0 1 2 3 4 5 6 7 . . .

random?

better experiment:

- generate random numbers
- use χ^2 test to check frequency values against uniform distribution

```
int k = 0;
while ( true ) {
  k = k*1664525 + 1013904223);
 System.out.print(k % V);
}
```

textbook algorithm that flunks χ^2 test

better hypotheses/experiments still needed

- many documented disasters
- active area of scientific research
- applications: simulation, cryptography
- connects to core issues in theory of computation

Warmup (continued)

Q. Is a given sequence of numbers random?

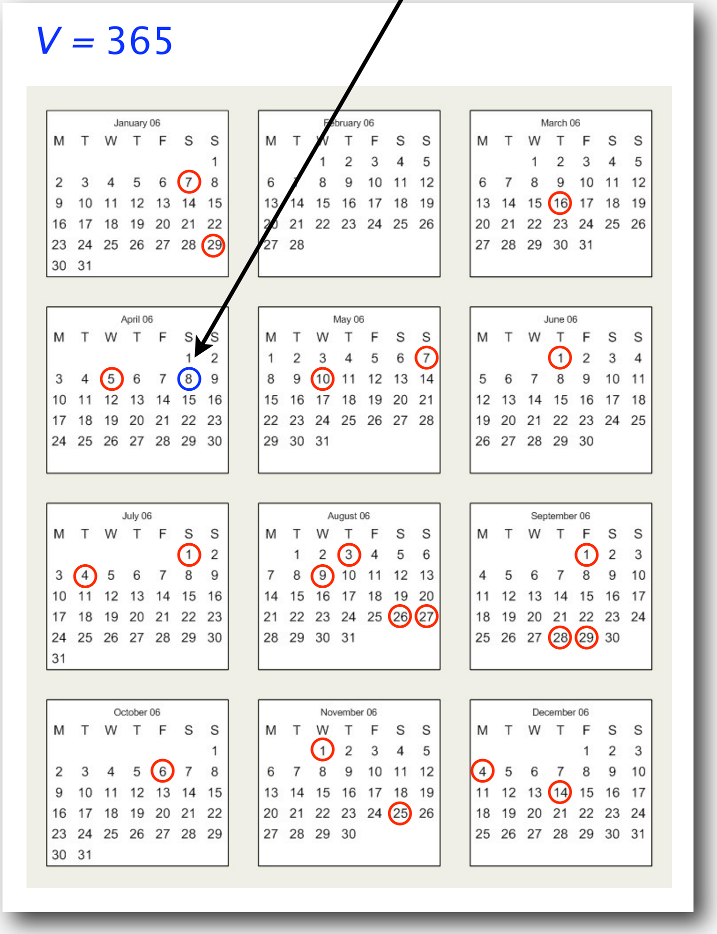
A. No.

Q. Does a given sequence exhibit some property that random number sequences exhibit?

Birthday paradox

Average count of random numbers generated until a duplicate happens is about $\sqrt{\pi V/2}$

average probes until duplicate is about 24



Example of a better experiment:

- generate numbers until duplicate
- check that count is close to $\sqrt{\pi V/2}$
- even better: repeat many times, check against distribution

"Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin" — John von Neumann

Finding an *st*-path in a graph

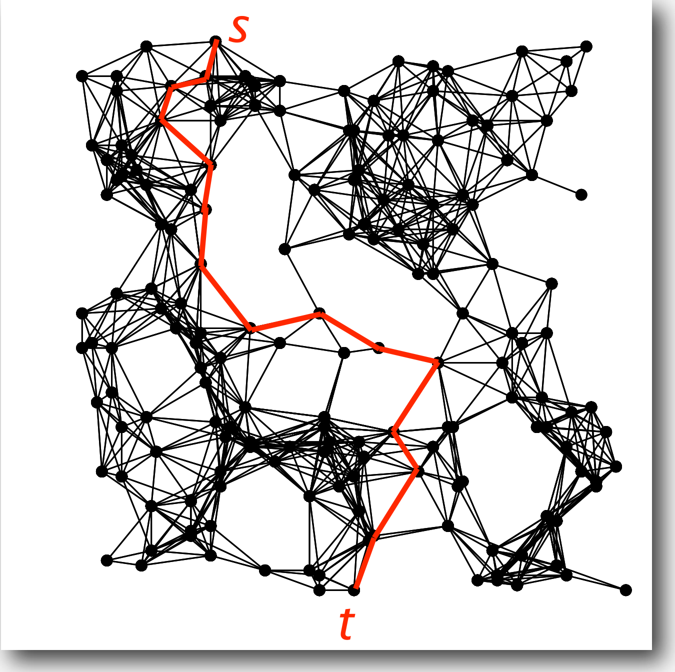
is a fundamental operation that demands understanding

Ground rules for this talk

- work in progress (more questions than answers)
- analysis of algorithms
- save "deep dive" for the right problem

Applications

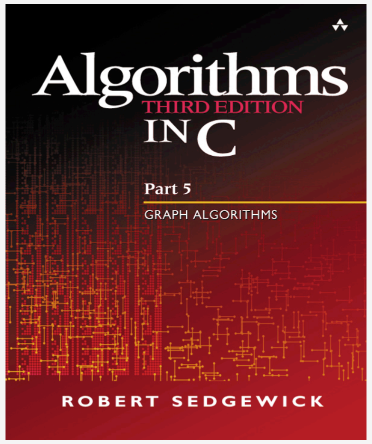
- graph-based optimization models
- networks
- percolation
- computer vision
- social networks
- (many more)



Basic research

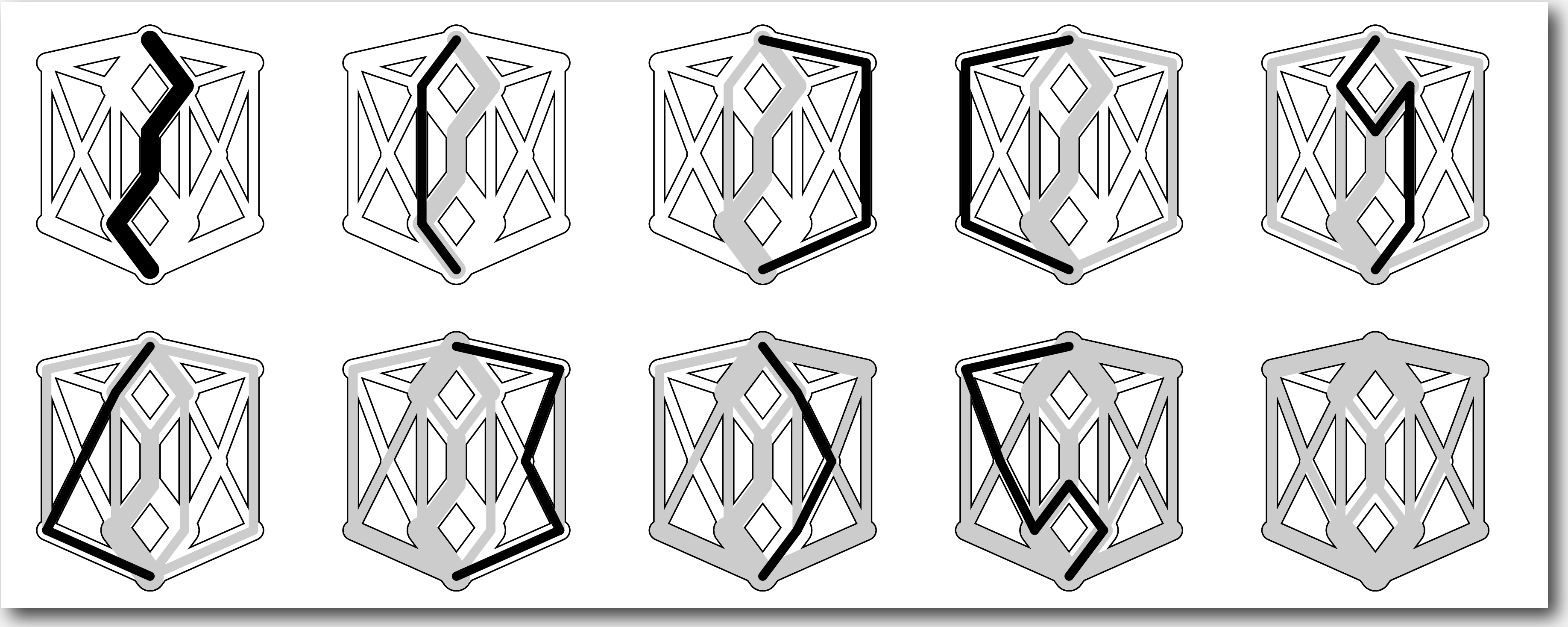
- fundamental abstract operation with numerous applications
- worth doing even if no immediate application
- resist temptation to prematurely study impact

Motivating example: maxflow



Ford-Fulkerson maxflow scheme

- find any s-t path in a (residual) graph
- augment flow along path (may create or delete edges)
- iterate until no path exists



Goal: compare performance of two basic implementations

- shortest augmenting path
- maximum capacity augmenting path

Key steps in analysis

- How many augmenting paths?
- What is the cost of finding each path?

research literature

this talk

Motivating example: max flow

Compare performance of Ford-Fulkerson implementations

- shortest augmenting path
- maximum-capacity augmenting path

Graph parameters

- number of vertices V
- number of edges E
- maximum capacity C

How many augmenting paths?

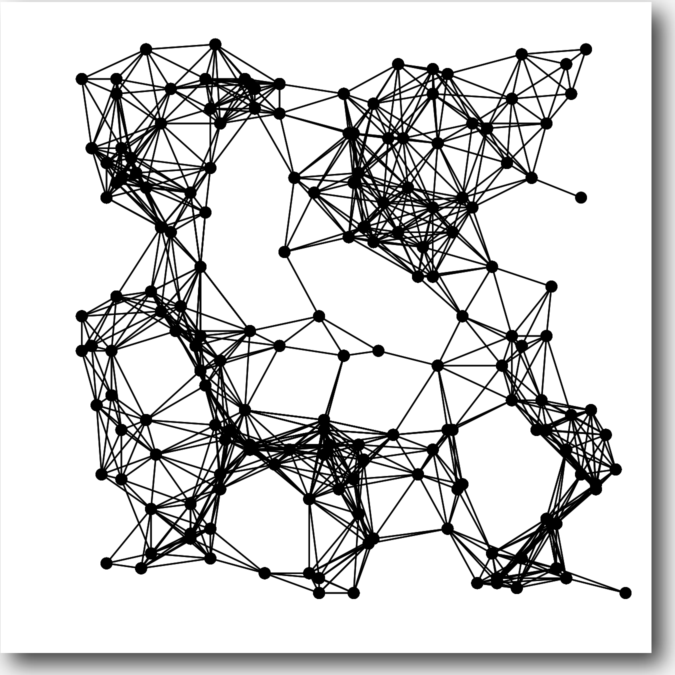
	worst case upper bound
shortest	VE/2 VC
max capacity	2E lg C

How many steps to find each path? E (worst-case upper bound)

Motivating example: max flow

Compare performance of Ford-Fulkerson implementations

- shortest augmenting path
- maximum-capacity augmenting path



Graph parameters for example graph

- number of vertices V = 177
- number of edges E = 2000
- maximum capacity C = 100

How many augmenting paths?

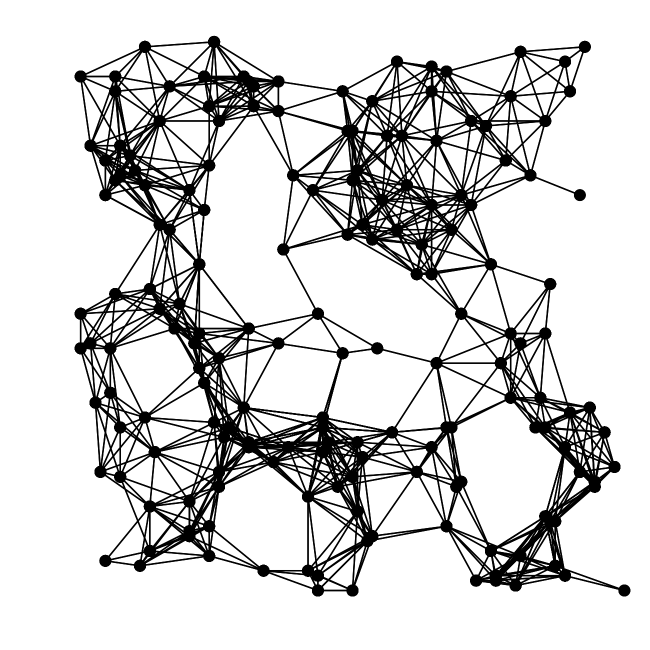
	worst case upper bound	*for example*
shortest	VE/2 VC	177,000 17,700
max capacity	2E lg C	26,575

How many steps to find each path? 2000 (worst-case upper bound)

Motivating example: max flow

Compare performance of Ford-Fulkerson implementations

- shortest augmenting path
- maximum-capacity augmenting path



Graph parameters for example graph

- number of vertices V = 177
- number of edges E = 2000
- maximum capacity C = 100

How many augmenting paths?

	worst case upper bound	*for example*	*actual*
shortest	VE/2 VC	177,000 17,700	37
max capacity	2E lg C	26,575	7

How many steps to find each path? < 20, on average

total is a factor of a million high for thousand-node graphs!

Motivating example: max flow

Compare performance of Ford-Fulkerson implementations

- shortest augmenting path
- maximum-capacity augmenting path

Graph parameters

- number of vertices V
- number of edges E
- maximum capacity C

Total number of steps?

	worst case upper bound
shortest	$VE^2/2$ VEC
max capacity	$2E^2 \lg C$

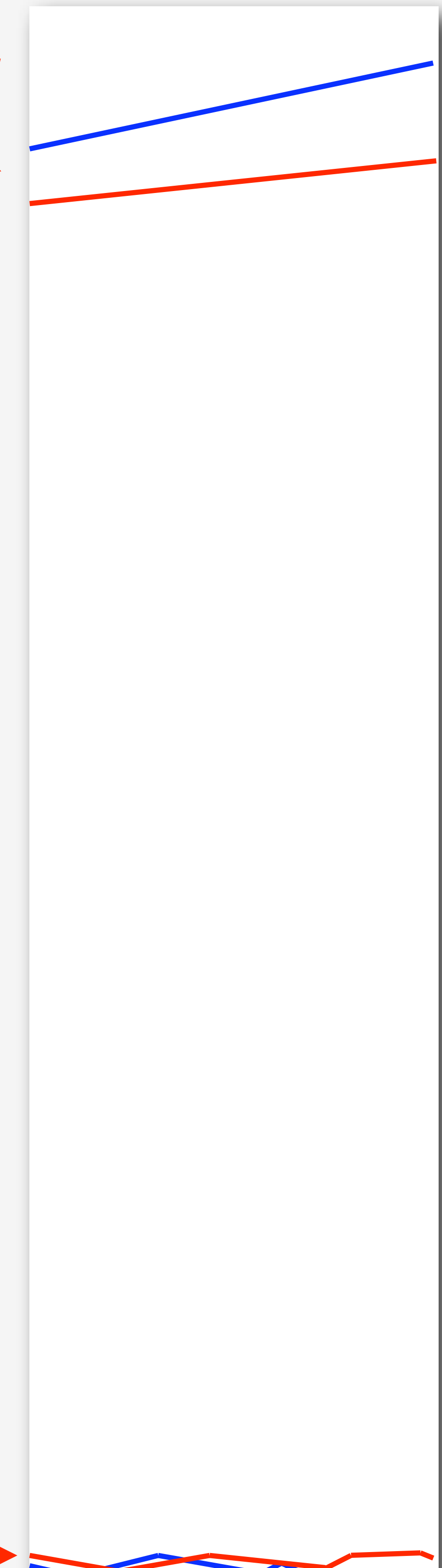
WARNING: The Algorithm General has determined that using such results to predict performance or to compare algorithms may be hazardous.

Motivating example: lessons

Goals of algorithm analysis

- predict performance (running time)
- guarantee that cost is below specified bounds

worst-case bounds



actual costs

Common wisdom

- random graph models are unrealistic
- average-case analysis of algorithms is too difficult
- worst-case performance bounds are the standard

Unfortunate truth about worst-case bounds

- often useless for prediction (fictional)
- often useless for guarantee (too high)
- often misused to compare algorithms

Bounds are useful in many applications:

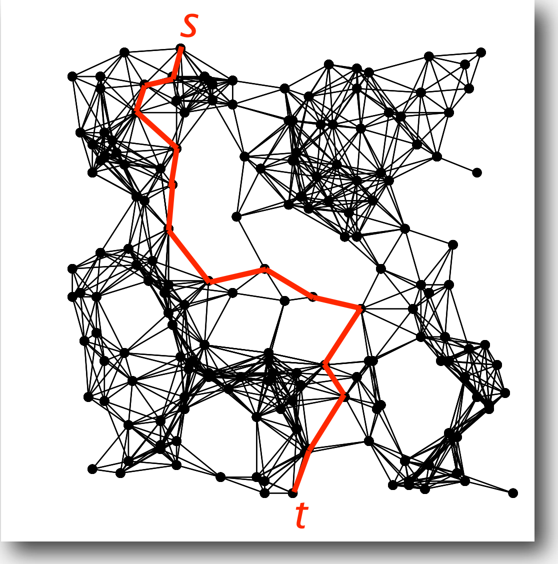
which ones??

Open problem: Do better!

Finding an *st*-path in a graph

is a basic operation in a great many applications

Q. What is the best way to find an *st*-path in a graph?



A. Several well-studied textbook algorithms are known

- Breadth-first search (BFS) finds the shortest path
- Depth-first search (DFS) is easy to implement
- Union-Find (UF) needs two passes

BUT

- all three process all E edges in the worst case
- diverse kinds of graphs are encountered in practice

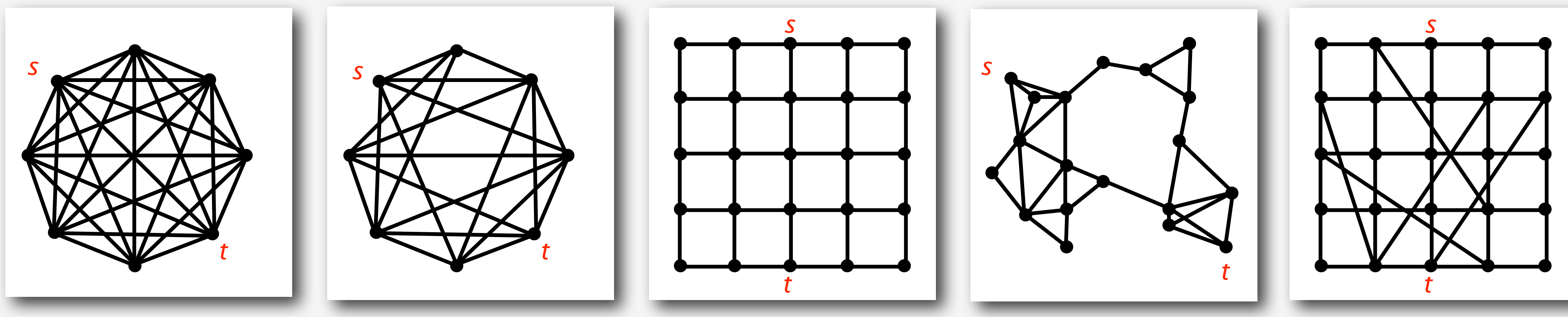
Worst-case analysis is useless for predicting performance

Which basic algorithm should a practitioner use?

Grid graphs

Algorithm performance depends on the graph model

complete *random* *grid* *neighbor* *small-world*



... (many appropriate candidates)

Initial choice: grid graphs

- sufficiently challenging to be interesting
- found in practice (or similar to graphs found in practice)
- scalable
- potential for analysis

if vertices have positions we can find short paths quickly with A (stay tuned)*

Ground rules

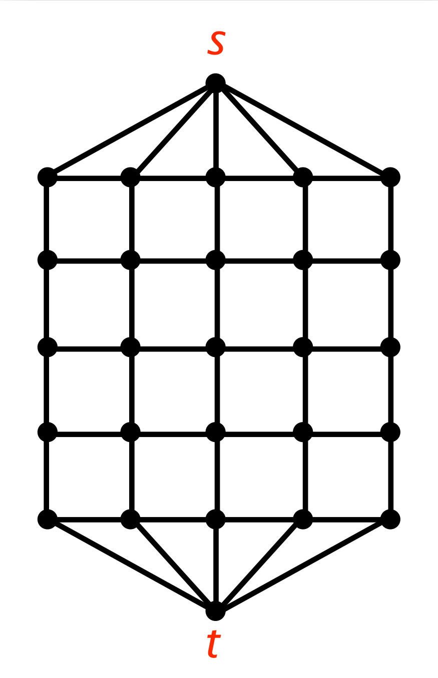
- algorithms should work for all graphs
- algorithms should not use any special properties of the model

Applications of grid graphs

conductivity
concrete
granular materials
porous media
polymers
forest fires
epidemics
Internet
resistor networks
evolution
social influence
Fermi paradox
fractal geometry
stereo vision
image restoration
object segmentation
scene reconstruction
.
.
.

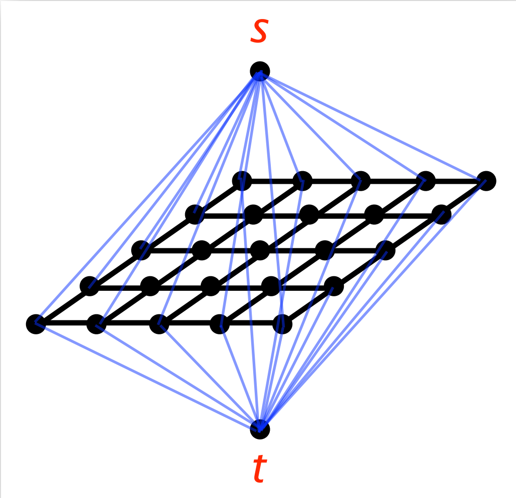
Example 1: Statistical physics

- percolation model
- extensive simulations
- some analytic results
- arbitrarily huge graphs



Example 2: Image processing

- model pixels in images
- maxflow/mincut
- energy minimization
- huge graphs



Literature on similar problems

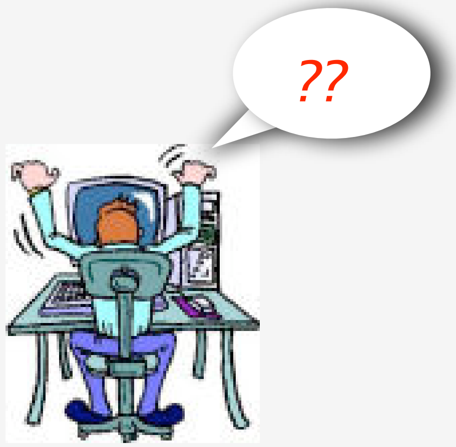
Percolation

Random walk

Nonselfintersecting paths in grids

Graph covering

Which basic algorithm should a practitioner use to find a path in a grid-like graph?



Finding an *st*-path in a grid graph

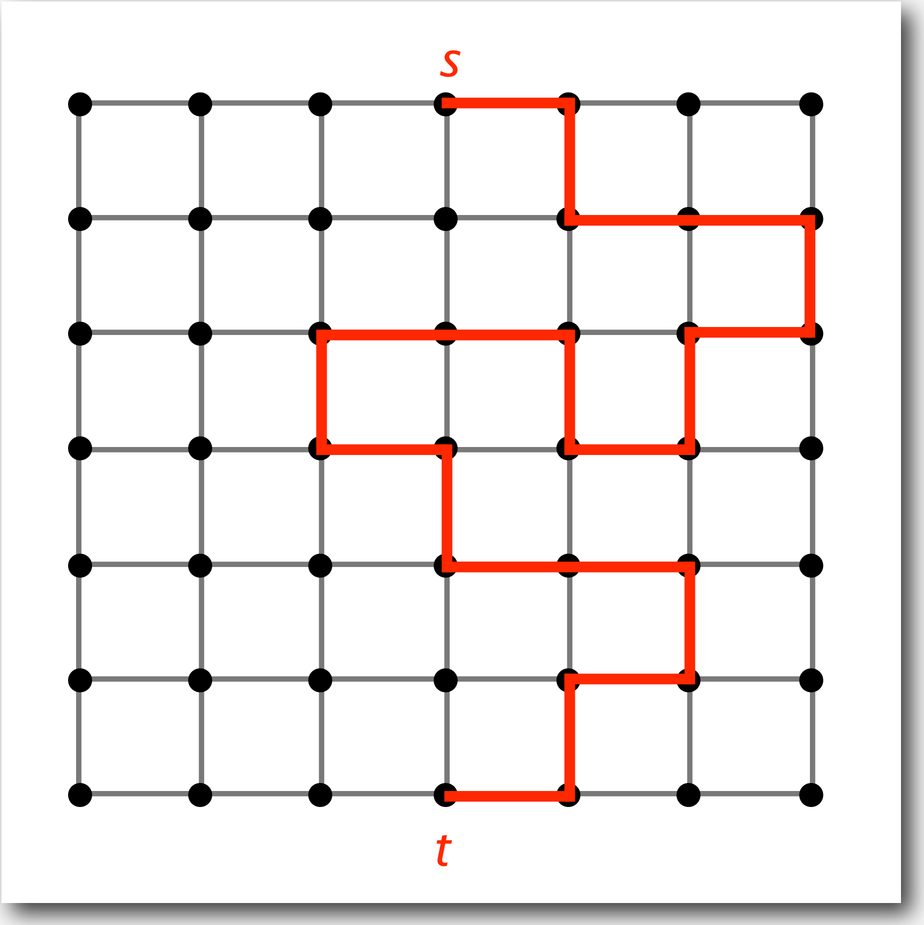
M by M grid of vertices

undirected edges connecting each vertex to its HV neighbors

source vertex *s* at center of top boundary

destination vertex *t* at center of bottom boundary

Find any path connecting *s* to *t*



M^2 vertices
about $2M^2$ edges

M	*vertices*	*edges*
7	*49*	*84*
15	*225*	*420*
31	*961*	*1860*
63	*3969*	*7812*
127	*16129*	*32004*
255	*65025*	*129540*
511	*261121*	*521220*

Cost measure: number of graph edges examined

Abstract data types

separate clients from implementations

A data type is a set of values and the operations performed on them

An abstract data type is a data type whose representation is hidden

Clients
invoke operations

Interface
specifies how to invoke ops

Implementations
code that implements ops

Implementation should not be tailored to particular client

Develop implementations that work properly for all clients
Study their performance for the client at hand

Graph abstract data type

Vertices are integers between 0 and V-1

Edges are vertex pairs

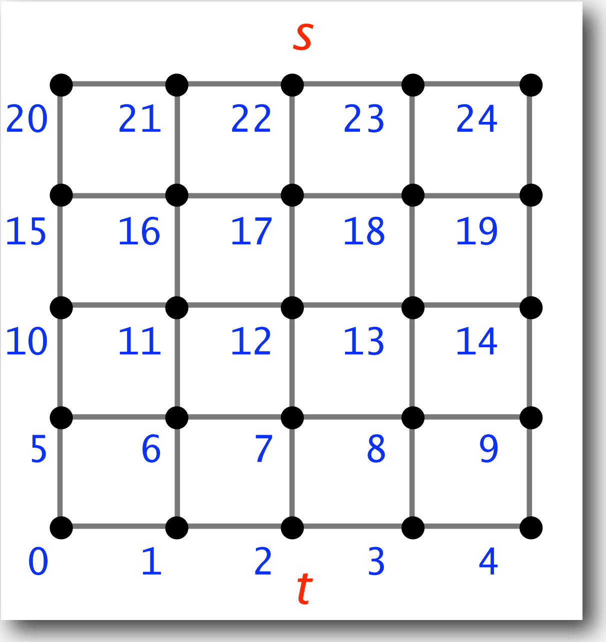
Graph ADT implements

- Graph(Edge[]) to construct graph from array of edges
- findPath(int, int) to conduct search from s to t
- st(int) to return predecessor of v on path found

Example: client code for grid graphs

```
int e = 0;
Edge[] a = new Edge[E];
for (int i = 0; i < V; i++)
 {  if (i < V-M) a[e++] = new Edge(i, i+M);
    if (i >= M) a[e++] = new Edge(i, i-M);
    if ((i+1) % M != 0) a[e++] = new Edge(i, i+1);
    if (i % M != 0) a[e++] = new Edge(i, i-1);
 }
GRAPH G = new GRAPH(a);
G.findPath(V-1-M/2, M/2);
for (int k = t; k != s; k = G.st(k))
  System.out.println(s + "-" + t);
```

M = 5



DFS: standard implementation

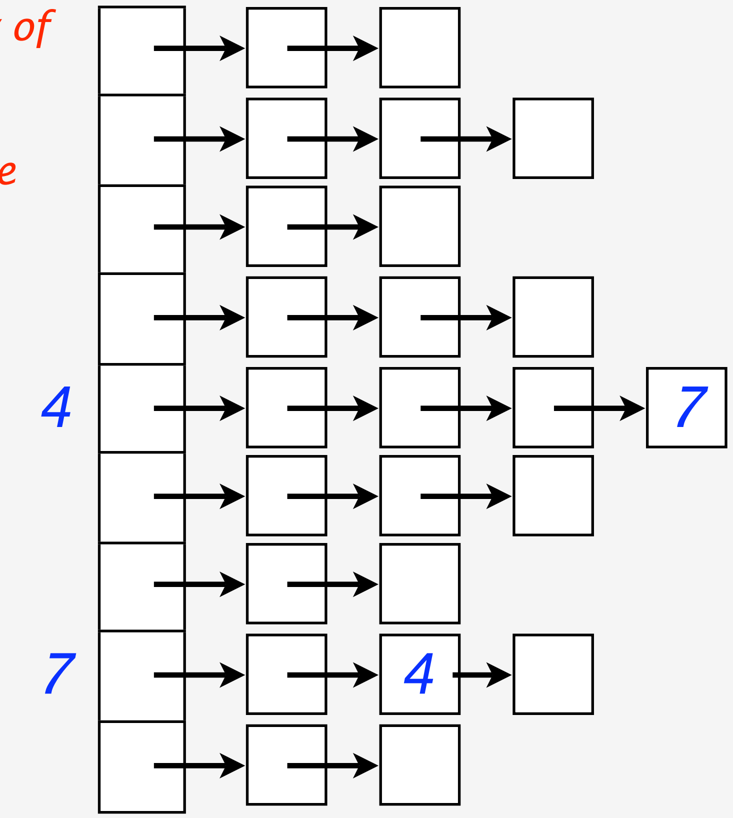
graph ADT constructor code

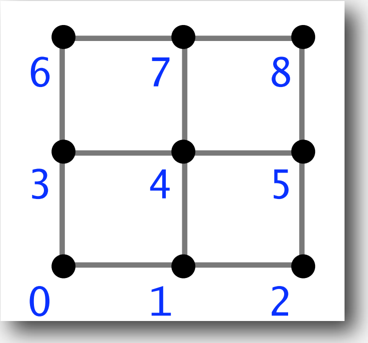
```
for (int k = 0; k < E; k++)
  { int v = a[k].v, w = a[k].w;
    adj[v] = new Node(w, adj[v]);
    adj[w] = new Node(v, adj[w]);
  }
```

graph representation

vertex-indexed array of linked lists

two nodes per edge





DFS implementation (code to save path omitted)

```
void findPathR(int s, int t)
  { if (s == t) return;
    visited(s) = true;
    for(Node x = adj[s]; x != null; x = x.next)
      if (!visited[x.v]) searchR(x.v, t);
  }
void findPath(int s, int t)
  { visited = new boolean[V];
    searchR(s, t);
  }
```

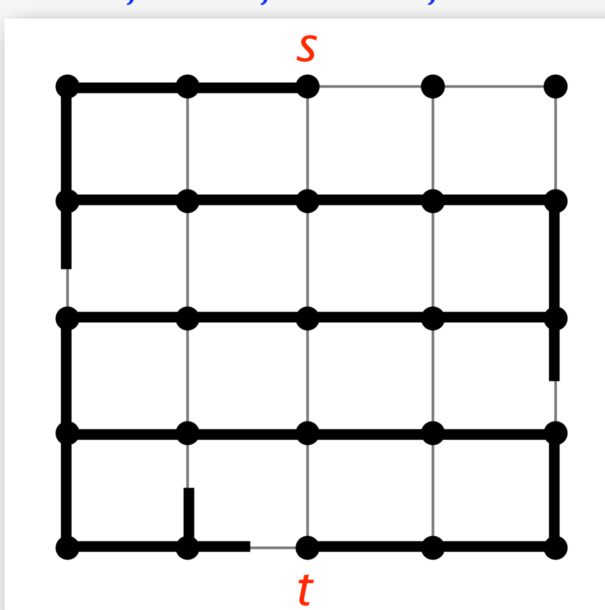
Basic flaw in standard DFS scheme

cost strongly depends on arbitrary decision in client code!

```
...
for (int i = 0; i < V; i++)
  {
    if ((i+1) % M != 0) a[e++] = new Edge(i, i+1);
    if (i % M != 0) a[e++] = new Edge(i, i-1);
    if (i < V-M) a[e++] = new Edge(i, i+M);
    if (i >= M) a[e++] = new Edge(i, i-M);
  }
...
```

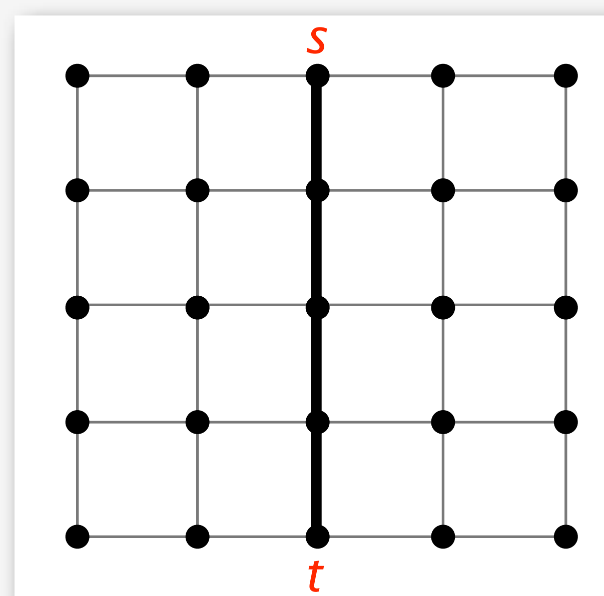
order of these statements determines order in lists

west, east, north, south



~E/2

south, north, east, west



~E $^{1/2}$

order in lists has drastic effect on running time

bad news for ANY graph model

Addressing the basic flaw

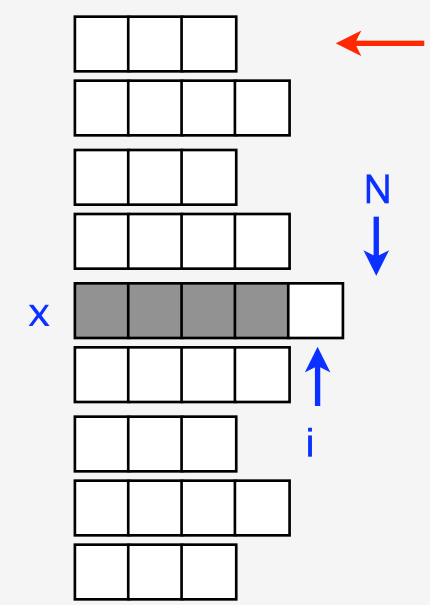
Advise the client to randomize the edges?

- no, very poor software engineering
- leads to nonrandom edge lists (!)

Randomize each edge list before use?

- no, may not need the whole list

Solution: Use a randomized iterator



represent graph with arrays, not lists

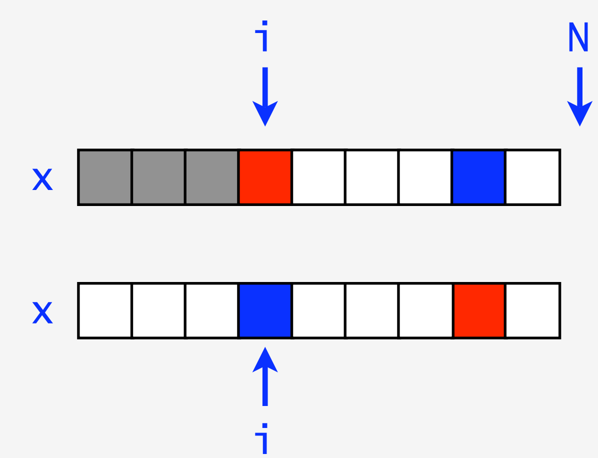
standard iterator

```
int N = adj[x].length;
for(int i = 0; i < N; i++)
  { process vertex adj[x][i]; }
```

randomized iterator

```
int N = adj[x].length;
for(int i = 0; i < N; i++)
  { exch(adj[x], i, i + (int) Math.random()*(N-i));
    process vertex adj[x][i];
  }
```

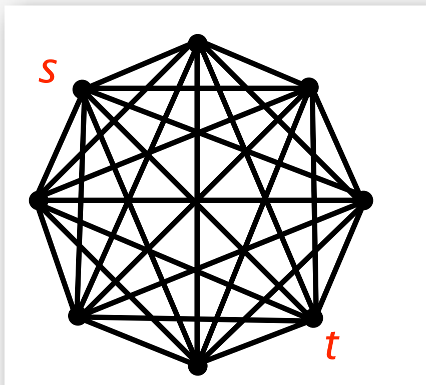
exchange random vertex from adj[x][i..N-1] *with* adj[x][i]



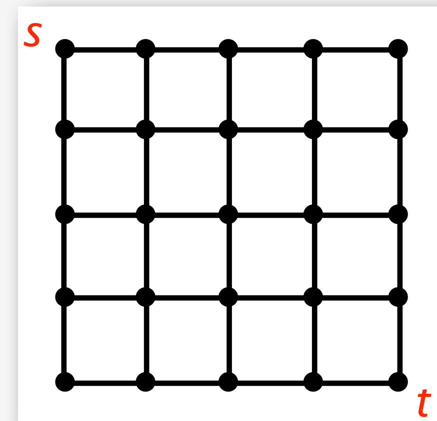
Use of randomized iterators

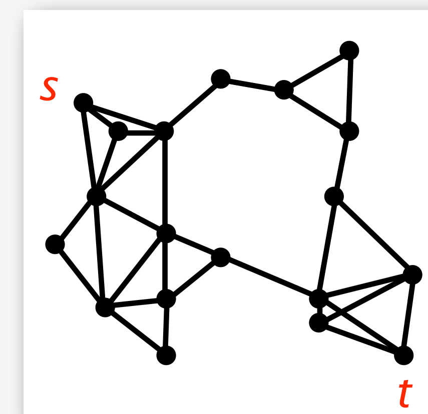
turns every graph algorithm into a randomized algorithm

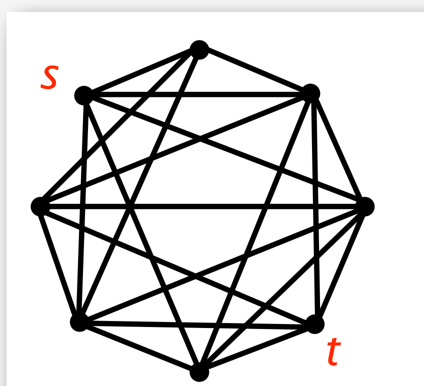
Important practical effect: stabilizes algorithm performance

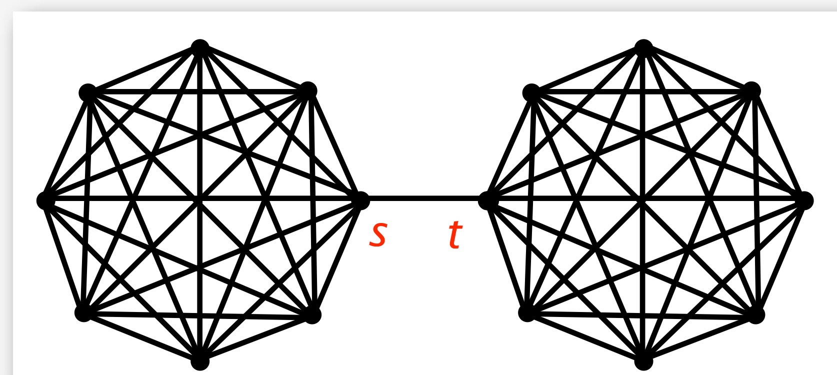


cost depends on problem
not its representation









Yields well-defined and fundamental analytic problems

- Average-case analysis of algorithm X for graph family Y(N)?
- Distributions?
- Full employment for algorithm analysts

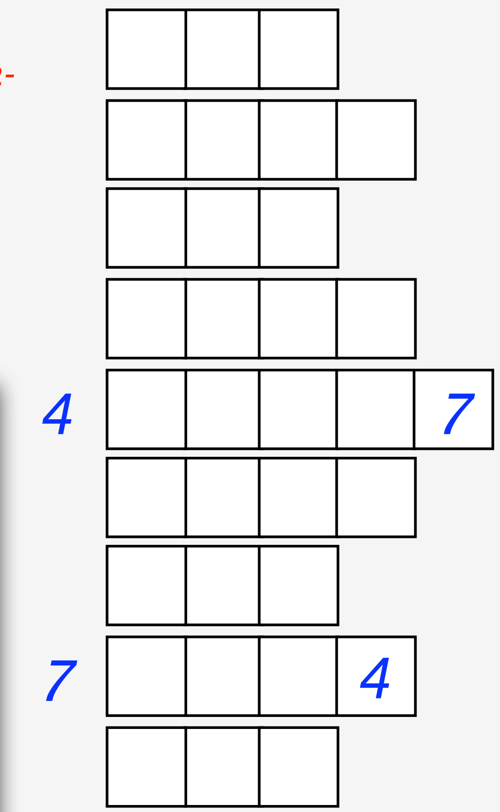
(Revised) standard DFS implementation

graph ADT constructor code

```
for (int k = 0; k < E; k++)
  { int v = a[k].v, w = a[k].w;
    adj[v][deg[v]++] = w;
    adj[w][deg[w]++] = v;
  }
```

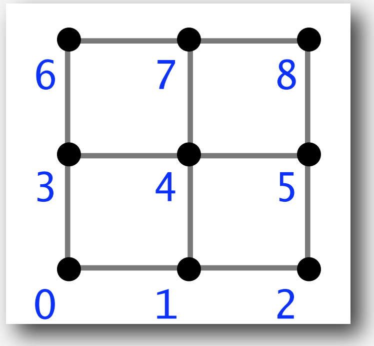
graph representation

vertex-indexed array of variable-length arrays



DFS implementation (code to save path omitted)

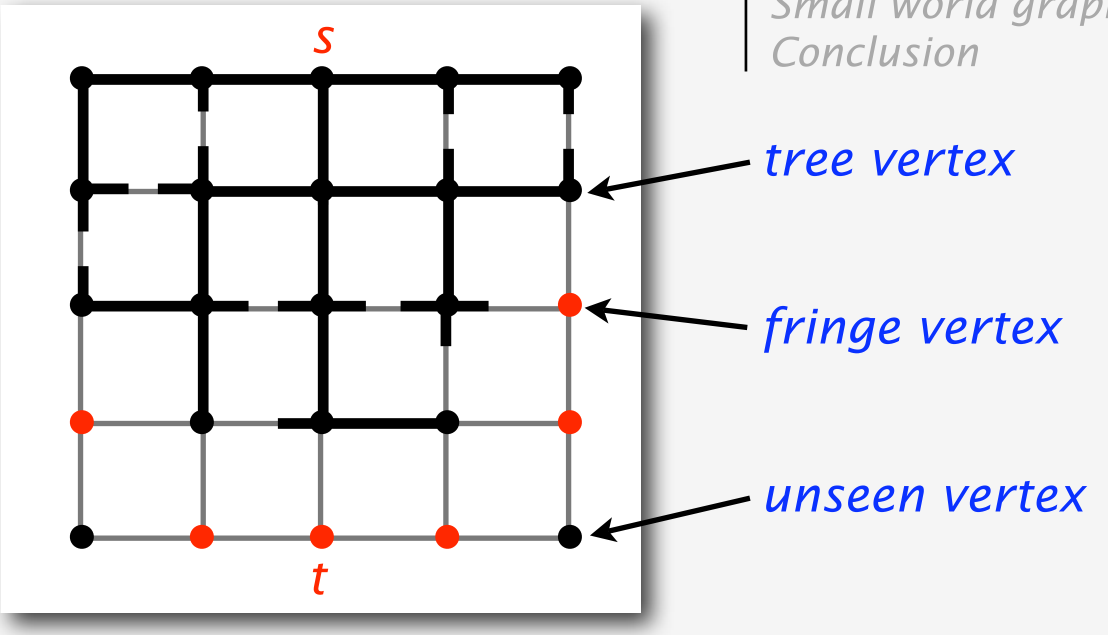
```
void findPathR(int s, int t)
  { int N = adj[s].length;
    if (s == t) return;
    visited(s) = true;
    for(int i = 0; i < N; i++)
    { int v = exch(adj[s], i, i+(int) Math.random()*(N-i));
      if (!visited[v]) searchR(v, t);
    }
  }
void findPath(int s, int t)
  { visited = new boolean[V];
    findpathR(s, t);
  }
```



BFS: standard implementation

Use a queue to hold fringe vertices

while Q is nonempty
get x from Q
done if x = t
for each unmarked v adj to x
put v on Q
mark v




```
void findPathaC(int s, int t)
 { Queue Q = new Queue();
   Q.put(s); visited[s] = true;
   while (!Q.empty())
     { int x = Q.get(); int N = adj[x].length;
       if (x == t) return;
       for (int i = 0; i < N; i++)
         { int v = exch(adj[x], i, i + (int) Math.random()*(N-i));
           if (!visited[v])
             { Q.put(v); visited[v] = true; }          }
     }
 }
```

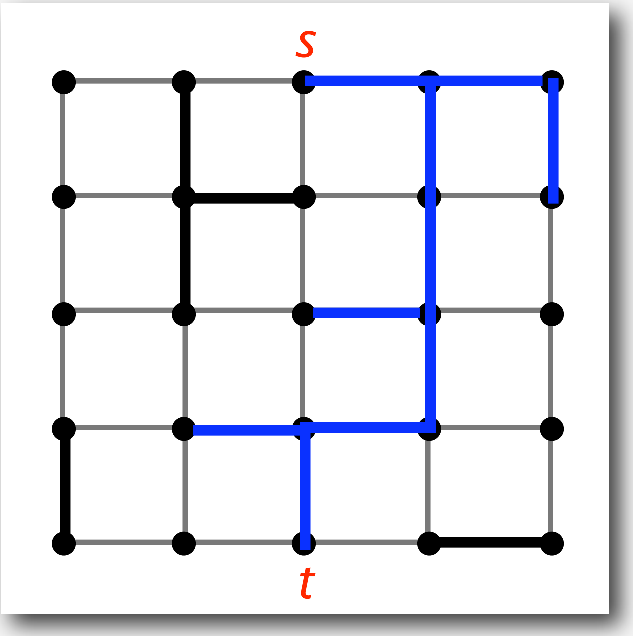
FIFO queue for BFS

randomized iterator

Generalized graph search: other queues yield A* and other graph-search algorithms

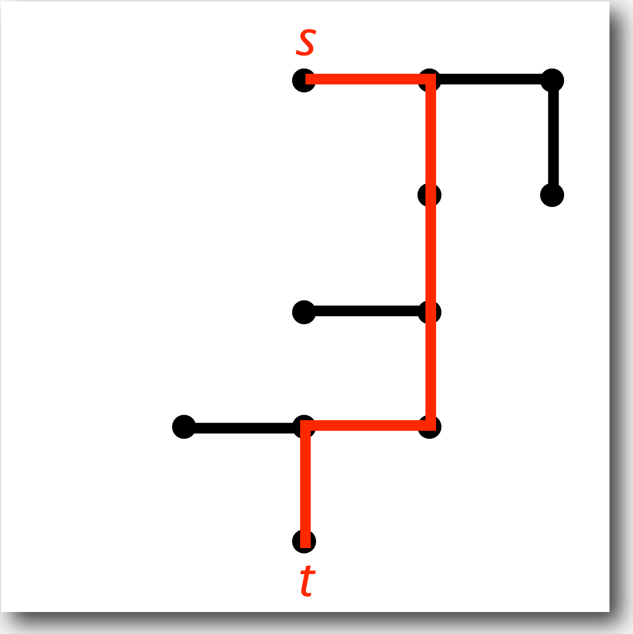
Union-Find implementation

1. Run union-find to find component containing *s* and *t*



initialize array of iterators
initialize UF array
while s and t not in same component
choose random iterator
choose random edge for union

2. Build subgraph with edges from that component
3. Use DFS to find *st*-path in that subgraph



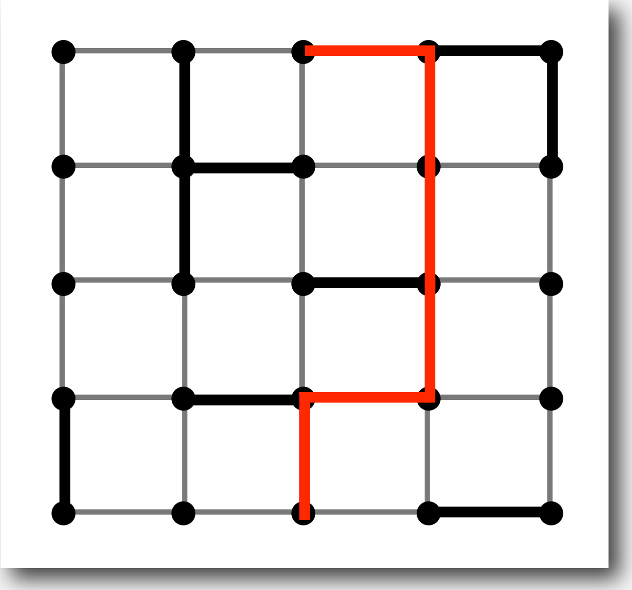
Experimental results

for basic algorithms

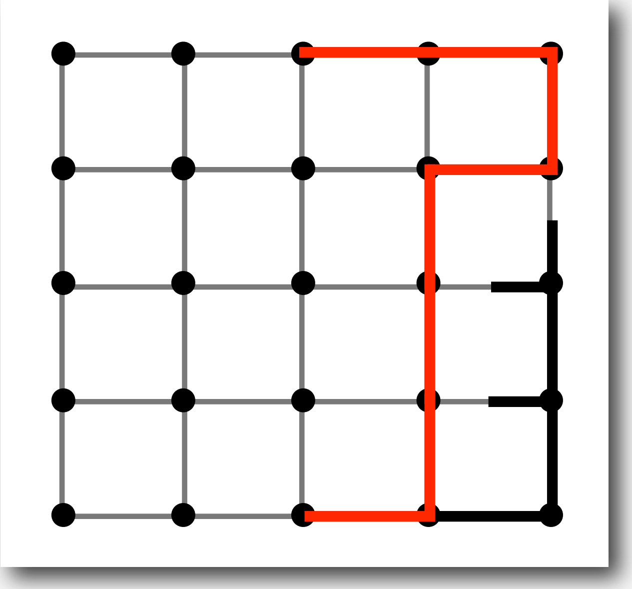
DFS is substantially faster than BFS and UF

M	V	E	BFS	DFS	UF
7	49	168	.75	.32	1.05
15	225	840	.75	.45	1.02
31	961	3720	.75	.36	1.14
63	3969	15624	.75	.32	1.05
127	16129	64008	.75	.40	.99
255	65025	259080	.75	.42	1.08

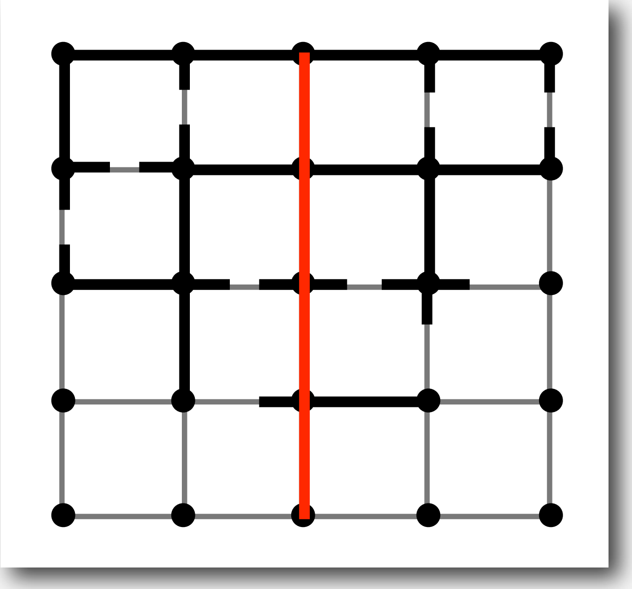
UF



DFS



BFS



Analytic proof?

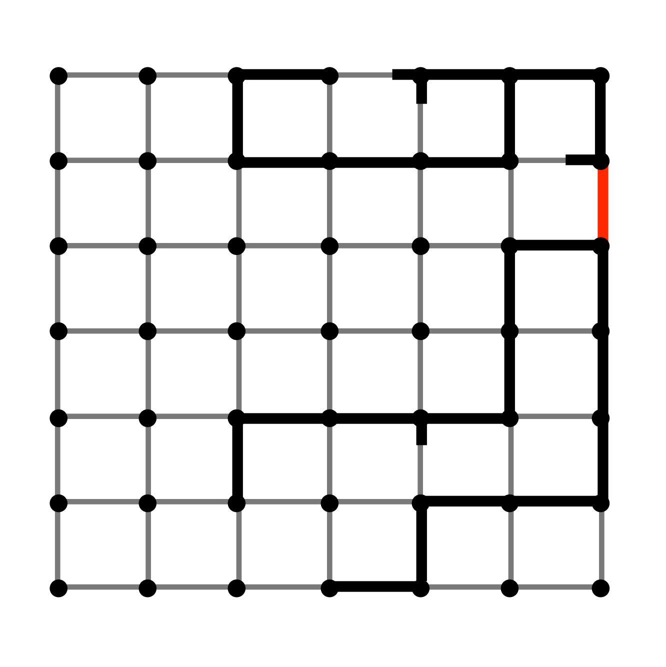
Faster algorithms available?

A faster algorithm

for finding an *st*-path in a graph

Use two depth-first searches

- one from the source *s*
- one from the destination *t*
- interleave the two



M	*V*	*E*	*BFS*	*DFS*	*UF*	*two*
7	49	168	.75	.32	1.05	.18
15	225	840	.75	.45	1.02	.13
31	961	3720	.75	.36	1.14	.15
63	3969	15624	.75	.32	1.05	.14
127	16129	64008	.75	.40	.99	.13
255	65025	259080	.75	.42	1.08	.12

Examines 13% of the edges

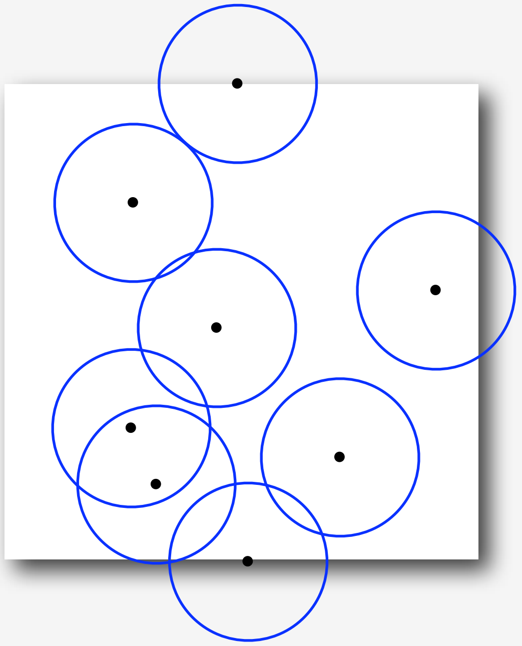
3-8 times faster than standard implementations

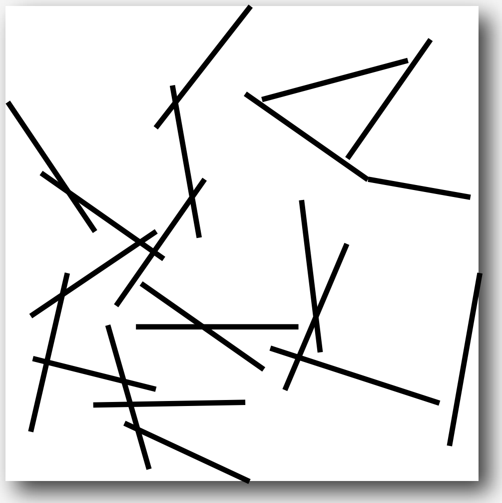
Not loglog E, but not bad!

Are other approaches faster?

Other search algorithms

- randomized?
- farthest-first?



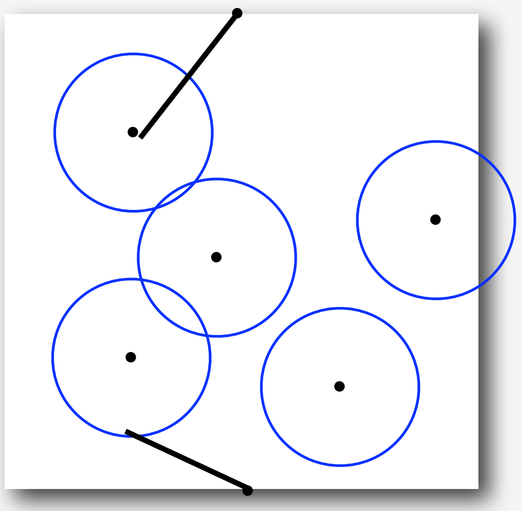


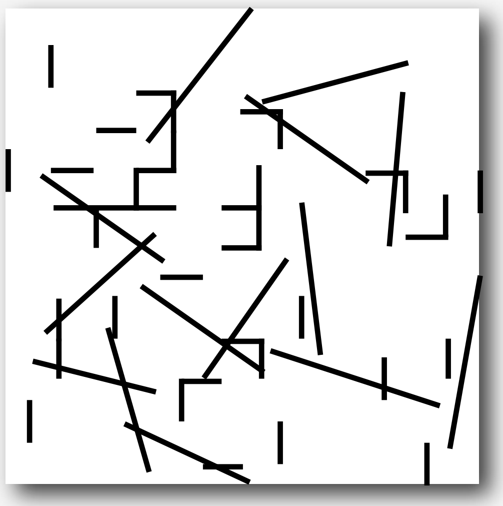
Multiple searches?

- interleaving strategy?
- merge strategy?
- how many?
- which algorithm?

Hybrid algorithms

- which combination?
- probabilistic restart?
- merge strategy?
- randomized choice?





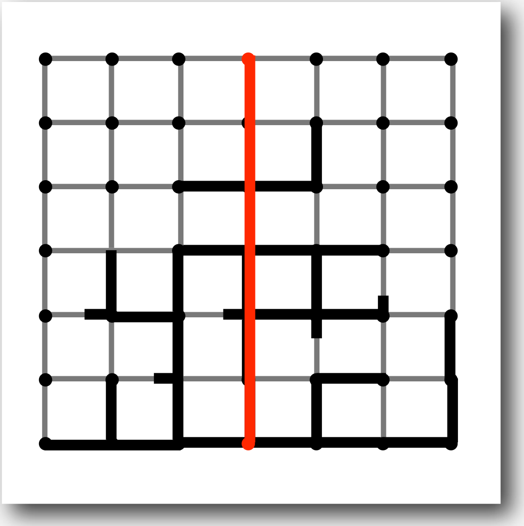
Better than constant-factor improvement possible? Proof?

Experiments with other approaches

Randomized search

- use random queue in BFS
- easy to implement

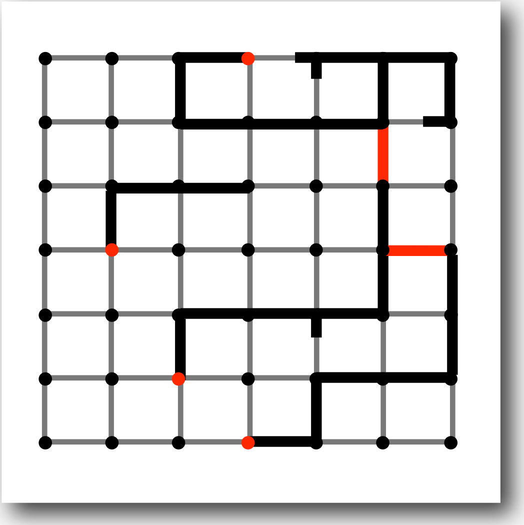
Result: not much different from BFS

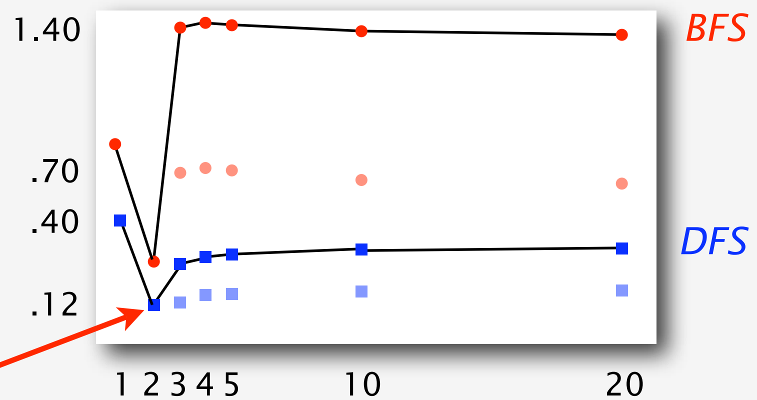


Multiple searchers

- use N searchers
- one from the source
- one from the destination
- N-2 from random vertices
- Additional factor of 2 for N>2

Result: not much help anyway





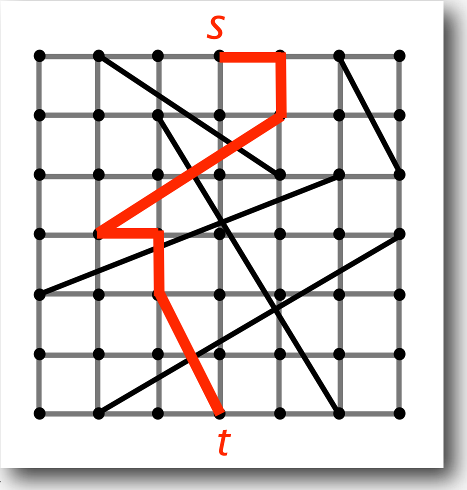
Best method found (by far): DFS with 2 searchers

Small-world graphs

are a widely studied graph model with many applications

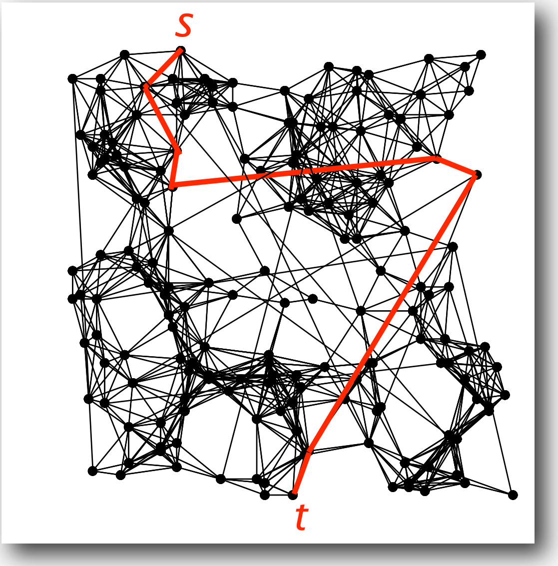
A small-world graph has

- large number of vertices
- low average vertex degree (sparse)
- low average path length
- local clustering



Examples:

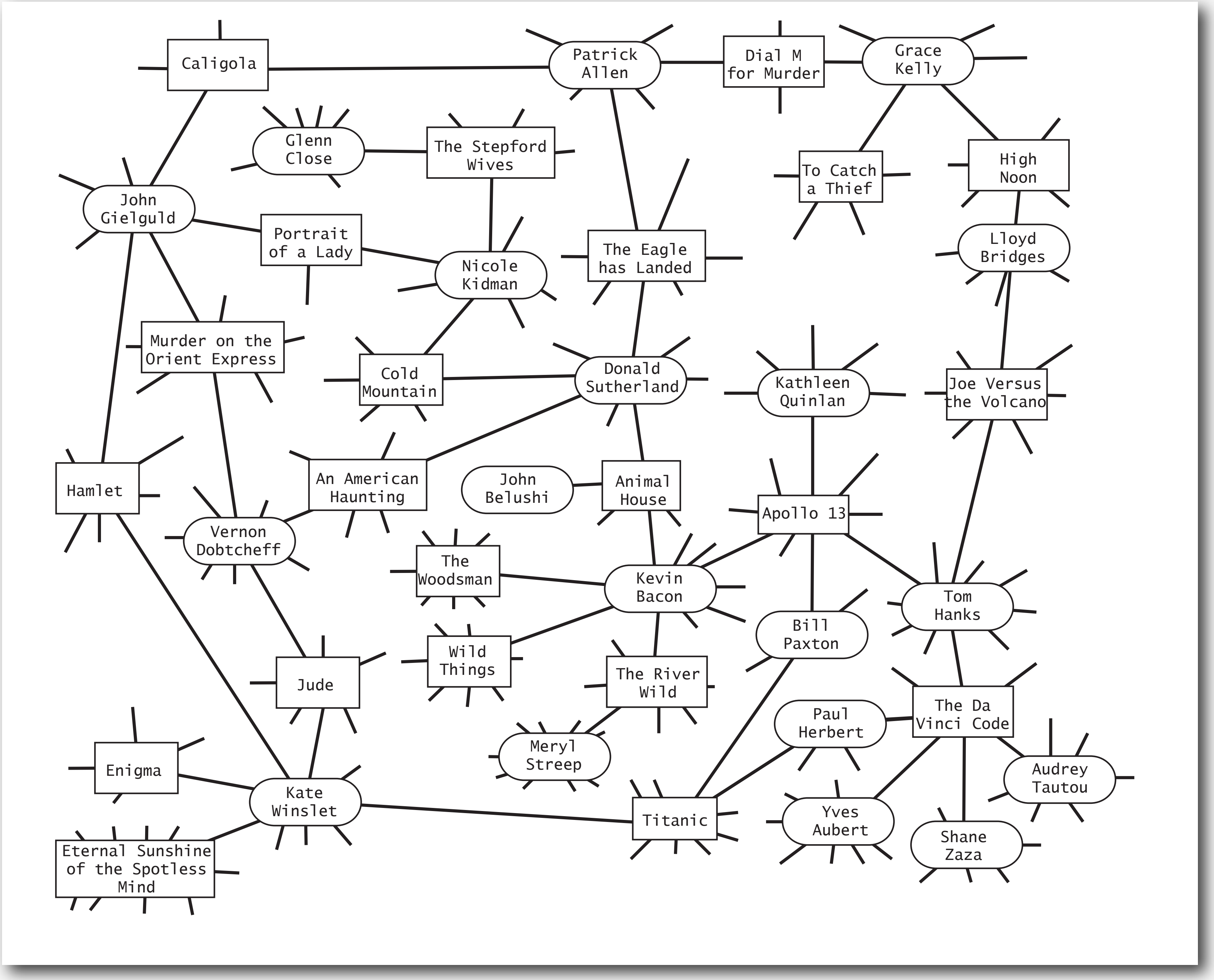
- Add random edges to grid graph
- Add random edges to any sparse graph with local clustering
- Many scientific models



Q. How do we find an *st*-path in a small-world graph?

Small-world graphs

model the six degrees of separation phenomenon



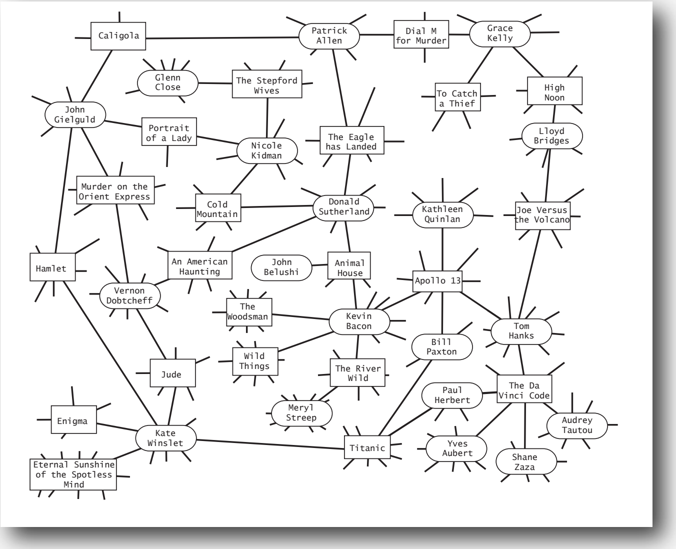
Example: Kevin Bacon number

Applications of small-world graphs

social networks
airlines
roads
neurobiology
evolution
social influence
protein interaction
percolation
internet
electric power grids
political trends
.
.
.

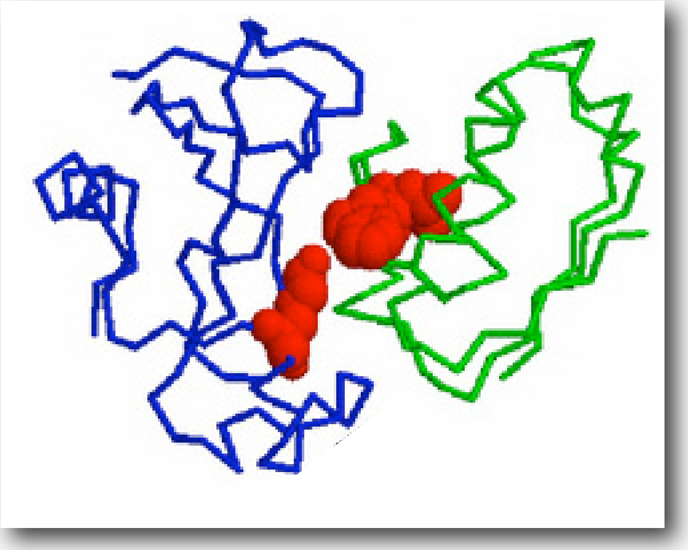
Example 1: Social networks

- infectious diseases
- extensive simulations
- some analytic results
- huge graphs



Example 2: Protein interaction

- small-world model
- natural process
- experimental validation



Finding a path in a small-world graph

is a heavily studied problem

Milgram experiment (1960)

Small-world graph models

- Random (many variants)
- Watts-Strogatz
- Kleinberg

add V random shortcuts to grid graphs and others

A uses ~ log E steps to find a path*

How does 2-way DFS do in this model?

no change at all in graph code just a different graph model

Experiment:

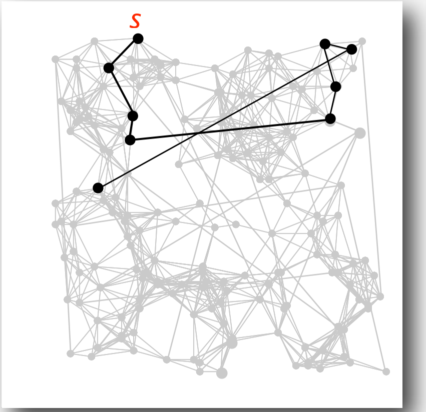
- add M ~ $E^{1/2}$ random edges to an M-by-M grid graph
- use 2-way DFS to find path

Surprising result: Finds short paths in ~ $E^{1/2}$ steps!

Finding a path in a small-world graph

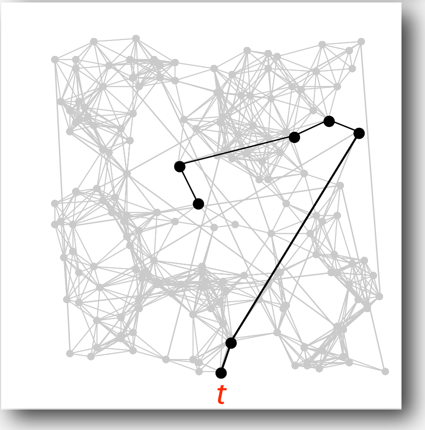
is much easier than finding a path in a grid graph

Conjecture: Two-way DFS finds a short *st*-path in sublinear time in any small-world graph

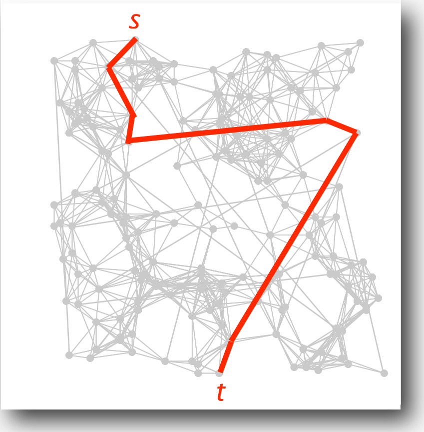


Evidence in favor

1. Experiments on many graphs
2. Proof sketch for grid graphs with V shortcuts
   - step 1: 2 $E^{1/2}$ steps ~ 2 $V^{1/2}$ random vertices
   - step 2: like birthday paradox



two sets of $2V^{1/2}$ randomly chosen vertices are highly unlikely to be disjoint



Path length?

Multiple searchers revisited?

Next steps: refine model, more experiments, detailed proofs

More questions than answers

Answers

- Randomization makes cost depend on graph, not representation.
- DFS is faster than BFS or UF for finding paths in grid graphs.
- Two DFSs are faster than 1 DFS — or N of them — in grid graphs.
- We can find short paths quickly in small-world graphs

Questions

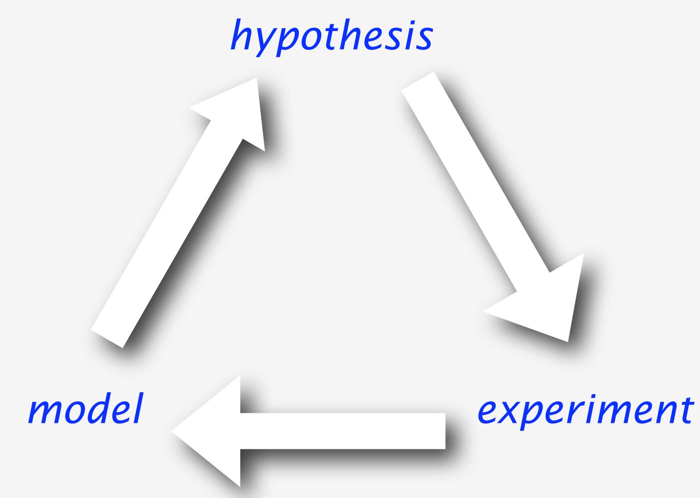
- What are the BFS, UF, and DFS constants in grid graphs?
- Is there a sublinear algorithm for grid graphs?
- Which methods adapt to directed graphs?
- Can we precisely analyze and quantify costs for small-world graphs?
- What is the cost distribution for DFS for any interesting graph family?
- How effective are these methods for other graph families?
- Do these methods lead to faster maxflow algorithms?
- How effective are these methods in practice?
- ...

Conclusion: subtext revisited

The scientific method is necessary
in algorithm design and implementation

Scientific method

- create a model describing natural world
- use model to develop hypotheses
- run experiments to validate hypotheses
- refine model and repeat



Algorithm designer who does not run experiments
risks becoming lost in abstraction

Software developer who ignores resource consumption
risks catastrophic consequences

We know much less than you might think
about most of the algorithms that we use