BELL'S PRIMENESS CRITERION AND THE SIMPLE LIE SUPERALGEBRAS

MARK C. WILSON

Abstract. We determine all finite-dimensional simple Lie superalgebras L such that $U(L)$ satisfies a primeness criterion due to Bell. Some open problems related to primeness of enveloping algebras are listed.

1. Introduction

This paper gives a complete account of the application of a primeness criterion, introduced in [Bel90], to the enveloping algebras of the Cartan type finite dimensional simple Lie superalgebras over a field of characteristic zero. It brings together recent work of the author and others, in the papers [Wil96], [WPW], [Wil] and [WP]. Combining these with the results of [Bel90] we obtain the following result.

Theorem. *Let L be a finite-dimensional simple Lie superalgebra over an algebraically closed field of characteristic zero. Then L satisfies Bell's criterion (so that $U(L)$ is prime) if and only if L is not of one of*: $b(n)$ for $n \geq 3$; $W(n)$ for odd $n \geq 5$; $S(n)$ for odd $n \geq 3$.

Of the exceptions above, $U(b(n))$ and $U(S(n))$ are not semiprime.

Those familiar with the material covered should still skim the introductory sections in order to fix notation. Section 2 introduces the Cartan type Lie superalgebras, and Section 3 contains the main work. Section 4 contains some comments and discusses some open problems.

Date. September 10, 1996.

1.1. Lie superalgebras. For definitions and all background, see [Kac77] and [Sch79]. There is no standard notation for many constructions; we follow mostly the notation of [Sch79].

Throughout K denotes an algebraically closed field of characteristic zero. A *Lie superalgebra* is a $\mathbb{Z}_2$-graded K-vector space $L = L_+ \dotplus L_-$ satisfying certain graded identities similar to those in a Lie algebra. We say L is *consistently* $\mathbb{Z}$-graded if L has a $\mathbb{Z}$-grading $L = \bigoplus_{n\in\mathbb{Z}} L_n$ as an algebra such that $L_+ = \sum_n L_{2n}, L_- = \sum_n L_{2n+1}$.

The finite-dimensional simple Lie superalgebras over K were classified in [Kac77]. There are two essentially different kinds of simple algebras. The *classical* algebras are those for which L_+ is reductive, or alternatively those for which L_- is a completely reducible L_+-module. The others belong to the 4 infinite families of *algebras of Cartan type*.

1.2. Primeness of enveloping algebras. The enveloping algebra of a Lie algebra is always a domain and hence prime. The usual proof is via a "filtered-graded" argument, with the standard (PBW) filtration of $U(L)$ being used. The direct analogue for Lie superalgebras fails since the associated graded ring is no longer prime. By using a different filtration, Allen Bell showed in [Bel90] that if L is a finite-dimensional Lie superalgebra over a field of characteristic zero, then the primeness of the universal enveloping algebra $U(L)$ is implied by the nonsingularity of the symmetric *product matrix* $([f_i, f_j])$. Here $\{f_1, \ldots, f_s\}$ is a basis for the odd part L_- of L, and the matrix is defined over the polynomial algebra $S(L_+)$. The condition is independent of which basis is chosen — it simply expresses the nonsingularity of a certain bilinear form.

Bell showed that his result applied to all of the classical simple Lie superalgebras except those of the family $b(n)$. The enveloping algebra of this outstanding case was shown not to be prime by a direct argument. An obvious next step is to consider the simple superalgebras of Cartan type. In the papers [Wil96], [WPW], [Wil] and [WP] the author (with the assistance of G. Pritchard and D.H. Wood) has determined whether or not Bell's criterion applies to these algebras. Since the answer is negative in some cases, and yet no information on whether Bell's criterion is necessary for primeness has so far been unearthed, the primeness or otherwise of the enveloping algebra of certain Cartan type algebras still remains open.

Note that if E is a subfield of K then $L \otimes_E K$ is a Lie superalgebra over K which satisfies Bell's criterion if and only if L does. Thus we are simultaneously checking Bell's criterion for all forms of the algebras considered here.

2. The Cartan type Lie superalgebras

The Cartan type simple (infinite-dimensional) Lie algebras all arise as subalgebras of the algebra of derivations of a finitely generated polynomial algebra. The analogue of a polynomial algebra in the superalgebra case is a Grassmann algebra (both are enveloping algebras of abelian Lie superalgebras). Since this is finite-dimensional so are all our Cartan type Lie superalgebras.

Let $n \geq 1$ be an integer and let V be an n-dimensional vector space over K. The Grassmann algebra $\Lambda = \Lambda(V)$ is the free anticommutative associative algebra on V, generated by V subject to the defining relations $vw + wv = 0$ for $v, w \in V$. One can also interpret Λ as the universal enveloping algebra of the odd abelian Lie superalgebra V. It has a consistent $\mathbb{Z}$-grading $\Lambda = \bigoplus_{r=0}^{n} \Lambda_r$ where $\Lambda_r = V^r$.

Let $N = \{1, \dots, n\}$ and fix a basis $\{v_i | i \in N\}$ of V. Given a subset I of N, order I as $i_1 < \cdots < i_r$ and form the element $v_I = v_{i_1} \cdots v_{i_r}$. The set of all such v_I, each I ordered arbitrarily, forms a basis for Λ, where we use the convention $v_\emptyset = 1$. Thus

$$\dim \Lambda_r = \binom{n}{r}.$$

The standard basis construction for Λ, that obtained from the PBW theorem, is to take all monomials v_I where I inherits its order from N. Of course the basis elements corresponding to a given I can differ only by a factor of ± 1 no matter which ordering is used.

The anticommutativity of Λ yields the obvious formula

$$v_I v_J = \begin{cases} \pm v_{I \cup J}, & I \cap J = \emptyset, \\ 0, & I \cap J \neq \emptyset. \end{cases} \tag{1}$$

2.1. $W(n)$**.** Take $V = K^n$ with standard basis $v_1, \dots, v_n$ and let $W = W(n) = D(\Lambda)$, the Lie superalgebra of superderivations of Λ. Then W is a graded subspace of the graded algebra $\mathrm{End}_K(\Lambda)$ and multiplication for homogeneous elements is given in the usual way by the supercommutator

$$[D_1, D_2] = \begin{cases} D_1 D_2 - D_2 D_1, & \text{if } D_1 \text{ or } D_2 \text{ is even}, \\ D_1 D_2 + D_2 D_1, & \text{if } D_1 \text{ and } D_2 \text{ are odd}. \end{cases}$$

The $\mathbb{Z}$-grading on W is consistent. Here the graded component W_r consists of all superderivations which map $V = \Lambda_1$ into Λ_{r+1}, so we have

$$W = \bigoplus_{r=-1}^{n-1} W_r.$$

Every element of W maps V into Λ and since it is a superderivation it is completely determined by its action on the generating subspace V. It follows that W can be identified with $\Lambda \otimes_K V^*$ as a vector space. Thus every element of W can be expressed as $\sum_i \lambda_i \partial_i$ where ∂_i is the odd superderivation $\partial/\partial v_i$. In particular for any ordering of the elements of the subsets I, the set of all $v_I \partial_i$, where $\partial_i = \partial/\partial_{v_i}$, forms a basis for W.

The tensor product of bases for Λ and V^* of course provides a basis for $\Lambda \otimes_K V^*$. We shall use a different basis in which the 2 tensor factors are not chosen independently. Let $\{v_1, \dots, v_n\}$ be a basis for V. Then we form a basis for W_r as follows.

For a pair (I, i) with $|I| = r + 1$, there are 2 possibilities. If $i \notin I$ say that (I, i) is of type (I,r). We order I naturally as a subset of N. The span of the $v_I \partial_i$ thus obtained is denoted $W_r^{(1)}$. If $i \in I$ then say that (I, i) is of type (II,r). We write $I = I' \cup \{i\}$ where I' is ordered in the natural way as a subset of N and we decree that $I' < i$, so that i is the last element of I. The span of the $v_I \partial_i$ thus obtained is denoted $W_r^{(2)}$. It follows that

$$\begin{aligned}
\dim W_r^{(1)} &= (n - r - 1)\binom{n}{r+1} \\
\dim W_r^{(2)} &= (r + 1)\binom{n}{r+1} \\
\dim W_r &= n\binom{n}{r+1}.
\end{aligned}$$

The multiplication formula for *odd* elements is

$$[v_I \partial_i, v_J \partial_j] = v_I \partial_i(v_J)\partial_j + v_J \partial_j(v_I)\partial_i. \tag{2}$$

Note that it is immediate from this formula that the product is zero if $|I \cap J| \geq 2$ or both $i \notin J$ and $j \notin I$ hold.

2.2. $S(n)$ **and** $\widetilde{S}(n)$**.** Every element of $W(n)$ can be uniquely expressed as $\sum_{i=1}^n \lambda_i \partial_i$ where $\lambda_i \in \Lambda(n)$. The kernel of the divergence mapping $\sum_{i=1}^n \lambda_i \partial_i \mapsto \sum_{i=1}^n \partial_i(\lambda_i)$ is a simple subalgebra of $W(n)$ called $S(n)$, which we often write as simply S.

S inherits a consistent $\mathbb{Z}$-grading

$$S = \bigoplus_{r=-1}^{n-2} S_r.$$

A spanning set for each S_r is as follows and contains 2 distinct types of elements. Those basis elements of type (I,r) are all those of the form $v_I\partial_i$ with $i \notin I$ and $|I| = r+1$. Those of type (II,r) are of the form $v_A h_{ij}$ where $i, j \notin A$ and $|A| = r$. Here $h_{ij} = v_i\partial_i - v_j\partial_j$. The type I elements are all linearly independent, and their span $S_r^{(1)}$ is independent of the span $S_r^{(2)}$ of the type II elements.

The type II elements are not independent however, since $h_{ij} + h_{jk} = h_{ik}$. We reduce the set of type II elements to a basis for $S_r^{(2)}$ as follows. For each A with $|A| = r$, order the complement $B = N\backslash A$ in the natural way as a subset of N and let i be the first element of B. Select those elements of the form $v_A h_{ij}$ where $i < j \in B$. These are easily seen to be independent and span $S_r^{(2)}$. The proof is the same as the proof that the standard basis for the Lie algebra $sl(n)$ is indeed a basis. In fact, as the restriction of the isomorphism $W_0 \cong gl(n)$ carries S_0 onto $sl(n)$, that situation is covered here. Under this last isomorphism the type (I,0) basis elements $v_i\partial_j$ correspond to the off-diagonal matrix units e_{ij} and the type (II,0) basis elements h_{1j} to the diagonal elements $e_{11} - e_{jj}$. We have

$$\begin{aligned} \dim S_r^{(1)} &= (n-r-1)\binom{n}{r+1} \\ \dim S_r^{(2)} &= (n-r-1)\binom{n}{r} \\ \dim S_r &= (n-r-1)\binom{n+1}{r+1}. \end{aligned}$$

When n is even the closely related algebra $\widetilde{S}(n)$ is defined as follows. For $-1 < r \leq n-2$, $\widetilde{S}_r = S_r$. Also $\widetilde{S}_{-1}$ is spanned by all $(1 + v_N)\partial_i$. Then $\widetilde{S}(n)$ is a simple Lie superalgebra not isomorphic to any $S(m)$. It does not inherit a consistent $\mathbb{Z}$-grading from W.

2.3. $H(n)$ **and** $\widetilde{H}(n)$**.** The subspace of $S(n)$ spanned by all superderivations of the form

$$D_\lambda = \sum_{i \in N} \partial_i(\lambda)\partial_i,$$

where $\lambda \in \Lambda$, is a Lie superalgebra called $\widetilde{H} = \widetilde{H}(n)$. $\widetilde{H}$ inherits a consistent $\mathbb{Z}$-grading from W and we have

$$\widetilde{H} = \bigoplus_{r=-1}^{n-2} H_r.$$

The subalgebra $H = H(n) = \bigoplus_{r=-1}^{n-3} H_r = [\widetilde{H}, \widetilde{H}]$ is a simple Lie superalgebra.

The homogeneous component H_r is isomorphic as a vector space (in fact as an H_0-module) to Λ_{r+2} via $D_\lambda \mapsto \lambda$, and so the derivations $x_I = D_{v_I}$, where $\emptyset \neq I \subseteq N$, form a basis for $\widetilde{H}$. Thus

$$\dim H_r = \binom{n}{r+2}.$$

Under the isomorphism $W_0 \cong gl(n)$ the $D_{v_{\{i,j\}}}$ are mapped to the standard basis elements $e_{ji} - e_{ij}$ for $H_0 \cong so(n)$.

3. Application of Bell's criterion

3.1. Generalities. For most of the classical simple Lie superalgebras, the weight space decomposition with respect to a Cartan subalgebra of L_+ has several nice properties. In particular if λ is a weight then so is $-\lambda$ and the associated weight spaces have the same dimension. This fails to hold for the algebras of Cartan type, and so the arguments in [Bel90] cannot be used.

The methods used below to show nonsingularity of the product matrix are varied. Essentially, we repeatedly combine specializations with block decompositions. The block decompositions arise sometimes from algebra gradings and sometimes from decomposition of the associated graph (see below). All of the Cartan type algebras have $\mathbb{Z}$-gradings with finite support, which yields a block structure to their product matrices. A further grading is obtained by considering the weight space decomposition of L considered as an L_0-module.

In every case the weight spaces corresponding to the highest even weights prove to be crucial. I have not found a satisfactory unified proof using this approach however, and so this remains a heuristic argument. The methods for showing singularity are even less systematic. In the case of $S(n)$ for n odd the zero pattern itself forces the product matrix to be singular, whereas for $W(n)$ with n odd more subtle (and interesting) methods are required.

The following elementary observation is used below. Let M be a matrix with block form

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

where B has more columns than rows. Then M is singular — in fact every term in the full expansion of the determinant is zero. The proof of the following proposition is immediate using this observation if we choose homogeneous bases for L ordered in the obvious way.

Proposition 3.1. *Let $L = \bigoplus_{i=-1}^{m} L_i$ be a finite-dimensional $\mathbb{Z}$-graded Lie superalgebra.*

(1) *Suppose that m is odd. Then a product matrix for L has the form*

$$\begin{pmatrix} L_{-1,-1} & L_{-1,1} & \dots & \dots & L_{-1,m} \\ L_{1,-1} & L_{1,1} & \dots & L_{1,m-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{m,-1} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

(a) *if $\dim L_a = \dim L_b$ whenever $a + b = m - 1$ then L satisfies Bell's criterion if and only if the product submatrices $L_{a,b}$ for such a and b are nonsingular.*

(b) *if $\dim L_{-1} < \dim L_m$ then L does not satisfy Bell's criterion.*

(2) *Suppose that m is even. Then a product matrix for L has the form*

$$\begin{pmatrix} L_{-1,-1} & L_{-1,1} & \dots & \dots & L_{-1,m-1} \\ L_{1,-1} & L_{1,1} & \dots & L_{1,m-3} & L_{1,m-1} \\ L_{3,-1} & L_{3,1} & \dots & L_{3,m-3} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{m-1,-1} & L_{m-1,1} & \dots & 0 & 0 \end{pmatrix}.$$

Thus if $\dim L_{-1} + \dim L_1 < \dim L_m$ then L does not satisfy Bell's criterion.

□

Unfortunately the last line of case (2) does not apply to any of the Cartan type algebras.

Another technique we use is to recast the problem in graph-theoretic language. Given a symmetric matrix M, there is a naturally associated graph $G(M)$ defined as follows. $G(M)$ has vertices labelled by the row indices and an edge from i to j if and only if the entry $M_{ij} \neq 0$. In other words, if we replace all nonzero elements of M by 1's then the resulting matrix is the adjacency matrix of $G(M)$. Finding a direct sum decomposition of M is equivalent in an obvious way to decomposing $G(M)$ into disjoint subgraphs.

When M results from a product matrix by specialization we shall say therefore that basis elements x, y are *linked* if the associated vertices are joined by an edge, i.e. $[x, y] \neq 0$. Often our basis elements are parametrized and we shall say that in this situation the relevant parameters are linked.

The following technical lemma will prove useful later. Here $A\Delta B$ denotes the symmetric difference $(A \cup B) \setminus (A \cap B)$.

Lemma 3.2. *Let N be a finite set of even size n. Let $X = \{x_S | \emptyset \subset S \subset N\}$ be a set of algebraically independent variables over $\mathbb{Z}_2$. Let r and s be odd positive integers with $r + s = n$. Then the matrix with rows indexed by all x_I with $|I| = r$ and all x_J with $|J| = s$, defined by*

$$M_{IJ} = \begin{cases} x_{I\Delta J}, & |I \cap J| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

is nonsingular over $\mathbb{Z}_2(X)$.

Proof. The matrix M is square, since it has dimensions $\binom{n}{r} \times \binom{n}{s}$. Note that if $|I \cap J| = 1$ then $|I\Delta J| = n-2$. Thus we may label the variables occurring in M by their ordered 2-element complements, for example $y_{13} = x_{N\setminus\{1,3\}}$.

We now make the specialization which sends y_{ij} to 0 unless $j - i = n/2$, and call the $n/2$ remaining variables $z_1 = y_{11'}, \ldots, z_{n/2}$. For each i let $i' = i + n/2 \pmod{n}$. Note that $(i')' = i$ and $z_i = z_{i'}$. The image of M under this specialization is a matrix whose only possibilities for nonzero entries are $\pm z_i$ for some i.

We shall obtain a block decomposition of M by decomposing the graph $G(M)$. If for simplicity we label the vertex corresponding to x_I by I, there is an edge in G joining I to J if and only if $[x_I, x_J] = \pm z_i$ for some i. We shall say that in this case I and J are joined by an edge of *colour* i.

Fix $i \in N$. It follows easily from the definition of M that there is an edge of colour i joining I and J if and only if $|I| + |J| = n$, one of i or i' belongs to both I and J and the other belongs to neither. Furthermore, for a given $I \neq N$, there is at most one edge of a given colour at the vertex I. Also there is at least one edge of some colour at the vertex I, since for some i we have $i \in I$ and $i' \notin I$.

We now obtain the promised block decomposition of M by showing that the set of colours occurring at a given vertex of $G(M)$ is constant on each component. To this end, we first show that vertices distance 2 apart have the same colours. Suppose that I and J are linked by an edge of colour i. Then without loss of generality $I \cap J = \{i\}$ and $I \cup J = N \setminus \{i'\}$. Let K be linked to J. If J and K are linked by an edge of colour j then either $\{i, i'\} = \{j, j'\}$, in which case $K = I$, or $\{i, i'\} \cap \{j, j'\} = \emptyset$. In the latter case we can assume $J \cap K = \{j\}$ and $J \cup K = N \setminus \{j'\}$. Thus $i' \in K$ since $i' \in J \cup K$ but $i' \notin J$. Let $X = J \cup \{i', j'\} \setminus \{i, j\}$. Then $|X| = |J|$, $K \cap X = \{i'\}, K \cup X = N \setminus \{i\}$ and so K and X are linked by an edge of colour i. Thus every colour occurring at I also occurs at K, and by symmetry I and K have the same colours.

It follows that if I and J are linked by an edge then they have the same colours, since if an edge of some colour i joins I and L, then J and L have the same colours and so the colour i occurs at J. By induction on the length of a path joining two vertices, the set of colours occurring at a vertex is constant on components. This decomposes $G(M)$ into a union of disjoint subgraphs, each corresponding to a given set of colours.

Now fix a block corresponding to a given set of colours. This matrix is such that in every row and column, each variable which is present occurs exactly once, perhaps with a minus sign. Then specializing all but one of these variables to zero we obtain a nonsingular monomial matrix. □

3.2. $W(n)$**.** We record here for later use some obvious but useful formulae. Here $p(I, i)$ denotes the position of the integer i in the ordered set I.

$$\partial_i(v_I) = \begin{cases} (-1)^{1+p(I,i)} v_{I\setminus\{i\}}, & i \in I, \\ 0, & i \notin I. \end{cases} \tag{3}$$

$$v_i\partial_i(v_I) = \begin{cases} v_I, & i \in I, \\ 0, & i \notin I. \end{cases} \tag{4}$$

$$(-1)^{1+p(I,i)} v_i v_{I\setminus\{i\}} = v_I = (-1)^{|I|-p(I,i)} v_{I\setminus\{i\}} v_i \qquad \text{if } i \in I. \tag{5}$$

3.2.1. *n even.*

Theorem 3.3. *Let $n \geq 4$ be even. Then $W(n)$ satisfies Bell's criterion and hence $U(W(n))$ is prime.*

Proof. Fix odd r, s with $-1 \le r, s \le n-1$ and $r+s = n-2$. Each block $W_{r,s}$ is square:

$$\dim W_r = \binom{n}{r+1} = \binom{n}{s+1} = \dim W_s \qquad \text{if } r+s = n-2,$$

so by Proposition 3.1 it suffices to show that each such block is nonsingular.

We first make the specialization which sends all type (II, $n-2$) variables $v_I \partial_i$ to zero. The remaining n variables are $x_k = v_{N\setminus\{k\}}\partial_k$, and we specialize these all to 1. Let M denote the resulting matrix; it suffices to prove that M is nonsingular.

We now calculate explicitly conditions on (I, i) and (J, j) which are equivalent to their being linked under this specialization.

The product in (2) is nonzero in W only if $i \in J$ or $j \in I$. Also it is clear that for the product to remain nonzero under our specialization it must lie in the span of x_i and x_j.

First suppose that $i = j$. The first term on the right side in the odd product formula (2) remains nonzero under our specialization if and only if $i \in J$ and $I \cup (J \setminus \{i\}) = N \setminus \{i\}$. Since $|I| + |J| = n$, this is equivalent to $I \cap J = \emptyset$ and $I \cup J = N$. Thus $i \notin I$. Similarly, if the second term remains nonzero then $i \in I$, $i \notin J$, $I \cap J = \emptyset$ and $I \cup J = N$. Hence at most one term on the right side of (2) remains nonzero, and the corresponding entry equals ± 1.

Now suppose that $i \ne j$. The first term in (2) remains nonzero if and only if $i \in J$ and $I \cup (J \setminus \{i\}) = N \setminus \{j\}$. This is equivalent to the conditions $I \cap J = \{i\}$, $I \cup J = N \setminus \{j\}$. Similarly the second term remains nonzero if and only if $j \in I$, $I \cap J = \{j\}$, $I \cup J = N \setminus \{i\}$. Note that again both terms cannot remain nonzero simultaneously and so the product in (2) specializes to 0 or ± 1.

Thus (I, i) and (J, j) are linked if and only if exactly one of the following conditions is satisfied:

(1) $i \in I, j \notin J, I \setminus \{i\}$ and $J \cup \{j\}$ are mutually complementary in N

(2) $i \notin I, j \in J, I \cup \{i\}$ and $J \setminus \{j\}$ are mutually complementary in N.

In each case the corresponding entry in M is just ± 1.

We now determine the components of $G(M)$ and thereby obtain a further block decomposition.

If $r = -1$ then all the (I, i) are of type I, and if $r = n-1$ all are of type II. Otherwise both types of variables occur. Now variables of the same type are not linked, and so for $1 \le r \le n-3$ the matrix M is, up to a reordering of rows and columns, the direct sum of two blocks.

Furthermore these blocks are square because

$$\dim W_r^{(1)} = (n-r-1)\binom{n}{r+1} = (s+1)\binom{n}{s+1} = \dim W_s^{(2)}.$$

The symmetry of the product matrix means that we need only consider the blocks formed by the product of type I by type II variables, so we assume $-1 \le r \le n-3$. We show that each such block M_1 is nonsingular, from which the theorem follows.

If (I, i) is of type (I,r) then by the above $A = I \cup \{i\}$ has size $r+2$, and $B = N \setminus A$ has size s. Conversely, given mutually complementary A and B with respective sizes $r+2$ and s, let i and j be elements of A. Then $(A \setminus \{i\}, i)$ and $(B \cup \{j\}, j)$ are linked and of type (I,r) and (II,s) respectively. It follows that each component of $G(M)$ consists of all the (I, i) and (J, j) determined by a given A. Thus after reordering rows and columns if necessary, each M_1 can be taken to be block diagonal, where the blocks have size $r+2$ and each block has every entry either 1 or -1.

Now we fix such a diagonal block X of size $r+2$. It suffices to prove that X is nonsingular. We first compute the entries of X.

For each A, $[v_{A\setminus i}\partial_i, v_B v_i \partial_i] = -v_{A\setminus i} v_B \partial_i$, whereas for $j \ne i$ we have $[v_{A\setminus i}\partial_i, v_B v_j \partial_j] = v_B v_{A\setminus i}\partial_i = v_{A\setminus i} v_B \partial_i$. Thus by reordering the rows or columns of X and multiplying columns or rows by -1 if necessary, we can arrange so that the only -1 entries occur along the leading diagonal and the other entries are all 1, i.e. X can be taken to be

$$\begin{pmatrix} -1 & 1 & \dots & 1 \\ 1 & -1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & -1 \end{pmatrix}.$$

Now it is well known (and straightforward to show) that such a matrix is nonsingular if its dimension is not 2×2. Since $r \ne 0$ (it is odd), X is nonsingular, and this concludes the proof. □

3.2.2. *n odd.* Case (2) of Proposition 3.1 applies here. Unfortunately since $\dim W_{-1} + \dim W_1 = n + n^2(n-1)/2 > n^2 = \dim W_{n-1}$ the last sentence there does not apply. We shall first estimate the rank of each block $W_{r,s}$ with $r + s = n - 1$. Note that necessarily $1 \le r, s \le n-2$.

The component W_{n-1} has basis consisting of all $z_k = v_N \partial_k$ with $k \in N$, so every nonzero entry in $W_{r,s}$ is a linear combination of the z_k.

Let $I, J \subseteq N$ with $|I| = r+1$, $|J| = n-r$. We now obtain conditions for (I, i) and (J, j) to be linked. It follows from (2) that a necessary condition for linking is that $I \cap J = \{i\}$ or $I \cap J = \{j\}$. These two possibilities are in fact mutually exclusive, since

(6) $$[v_I \partial_i, v_J \partial_i] = 0 \qquad \text{if } |I| \text{ and } |J| \text{ are even and } I \cap J = \{i\}.$$

To see this, we compute:

$$\begin{aligned} [v_I \partial_i, v_J \partial_i] &= -v_I v_{J\setminus\{i\}} \partial_i - v_J v_{I\setminus\{i\}} \partial_i \\ &= \left(-v_{I\setminus\{i\}} v_i v_{J\setminus\{i\}} - v_{J\setminus\{i\}} v_i v_{I\setminus\{i\}}\right) \partial_i \\ &= \left(v_{I\setminus\{i\}} v_{J\setminus\{i\}} v_i - v_{I\setminus\{i\}} v_{J\setminus\{i\}} v_i\right) \partial_i \\ &= 0. \end{aligned}$$

In summary, (I, i) and (J, j) are linked if and only if $i \neq j$ and either $I \cap J = \{i\}$ or $I \cap J = \{j\}$. The corresponding entry in M equals $\pm z_k$ for some $k \in N$.

We analyze the 2 types of variables separately.

If (I, i) is of type (I,r), then (I, i) links to (J, j) if and only if (J, j) is of type (II,s), and $I \cap J = \{j\}$. For each $j \in I$ there is exactly one such J and in fact we have $v_J \partial_j = v_{N\setminus I} v_j \partial_j$ by our basis convention. Thus the corresponding entry in the product matrix is $v_{N\setminus I} v_I \partial_i$. Note that this is independent of J and j and so a row indexed by such a pair (I, i) has precisely $|I|$ nonzero entries all of which are the same. Furthermore for a fixed I the nonzero entries occur in the same columns for all i.

If (I, i) is of type (II,r) then there are 3 subcases.

(i)$\mathbf{I \cap J = \{j\}}$. We have $v_J \partial_j = v_{N\setminus I} v_j \partial_j$ and $v_I \partial_i = v_{I\setminus\{i\}} v_i \partial_i$ and so the entry in the product matrix is $v_{N\setminus I} v_{I\setminus\{i\}} v_i \partial_i$.

(ii)$\mathbf{I \cap J = \{i\}, j \in J}$. Here $v_I \partial_i = v_{N\setminus J} v_i \partial_i$ and the corresponding entry is $v_{N\setminus J} v_{J\setminus\{j\}} v_j \partial_j$.

(iii)$\mathbf{I \cap J = \{i\}, j \notin J}$. Here the corresponding entry is $v_{N\setminus J} v_J \partial_j$.

We now estimate the rank of each block $W_{r,s}$.

Lemma 3.4. *The rank of* $W_{r,s}$ *is at most* $\binom{n+1}{r+1}$.

Proof. Fix $A \subseteq N$ with $|A| = r$. For a fixed $k \in B = N\setminus A$, consider the submatrix S_k of $W_{r,s}$ formed by all rows indexed by pairs $(A \cup \{k\}, i)$ as i ranges over B. By the analysis above, the columns which correspond to the nonzero entries in S_k are indexed by pairs of the 4 types

(1) $(B, j), j \in A$
(2) (B, k)
(3) $(B, j), j \in B \setminus \{k\}$
(4) $(B \setminus \{k\} \cup \{j\}, j), j \in A$.

Let F be the function field $= K(z_1, \dots, z_n)$. The rows where $i \neq k$ span a 1-dimensional F-subspace. Thus using suitable row operations over F we may assume that such rows contain only 1's and 0's. Furthermore the 1's occur precisely in the columns of the 2nd and 4th types above.

We now compute the remaining entries of S_k, namely those in the row with $i = k$. For the columns of the first type, case (iii) above applies and the entry is $v_A v_B \partial_j$. This is equal to $\epsilon(A) z_j$ where $\epsilon(A) = \pm 1$. For the column of the second type the entry is of course zero by equation (6). For the columns of the third type, case (ii) above applies and the entry is $v_A v_{B\setminus\{j\}} v_j \partial_j$. This can be rewritten as $(-1)^{|B|-p(B,j)} v_A v_B \partial_j$ and this is equal to $(-1)^{p(B,j)} v_A v_B \partial_j$ since $|B| = n-1-r$ is even. We can write this as $\varepsilon(A,j) z_j$ where $\varepsilon(A,j) = \pm 1$. Finally for columns of the fourth type, case (i) applies. The corresponding entry is $v_{B\setminus\{k\}} v_A v_k \partial_i$. This simplifies to $v_A v_k v_{B\setminus\{k\}} \partial_k$ by anticommutativity and this can further be rewritten as $(-1)^{1+p(B,k)} v_A v_B \partial_k$. In terms of the notation of the previous case this is equal to $-\epsilon(A,k) z_k$.

Thus S_k may be represented by the following table. Here to save space we write $B_k = B \setminus \{k\}$ and $B_{kj} = (B \setminus \{k\}) \cup \{j\}$.

	$(B,j), j \in A$	(B,k)	$(B,j), j \in B_k$	$(B_{kj},j), j \in A$
$i = k:$	$\epsilon(A) z_j$	0	$\epsilon(A,j) z_j$	$-\epsilon(A,k) z_k$
$i \neq k:$	0	1	0	1

The first row of the table represents one row of S_k whereas the second row represents s rows. Similarly, each column of the table may represent many columns of S_k.

By adding $\epsilon(A,k) z_k$ times any of the rows with $i \neq k$ to the row with $i = k$ we convert S_k to a matrix which may be represented by the following table.

	$(B,j), j \in A$	$(B,j), j \in B$	$(B_{kj},j), j \in A$
$i = k:$	$\epsilon(A) z_j$	$\epsilon(A,j) z_j$	0
$i \neq k:$	0	δ_{kj}	1

In particular, note that if we keep A fixed and perform the above procedure for each $k \in B$ in turn, all the rows with $i = k$ are now identical, so form a rank 1 submatrix.

Now allow A to vary. Each row of $W_{r,s}$ which is indexed by some (I,i) of type (II,r) appears precisely once in the above construction. Thus the total contribution to the rank of $W_{r,s}$ by such rows is at most equal to the number of A, namely $\binom{n}{r}$. The total contribution to the rank by rows of type (I,r) is at most equal to $\binom{n}{r+1}$. Thus $W_{r,s}$ has rank at most $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$. □

The main result follows directly:

Theorem 3.5. *If n is odd, $W(n)$ satisfies Bell's criterion only for $n = 3$.*

Proof. The case $n = 1$ is trivial and the associated 1×1 product matrix is zero. Now assume that $n \geq 3$. The submatrix $W_{\cdot,n-2}$ (the rightmost "column" of the product matrix) consists of 2 nonzero blocks and has dimensions $(n2^{n-1}) \times n^2$. It follows from Lemma 3.4 that the rank of $W_{\cdot,n-2}$ is at most $n + \binom{n+1}{2} = n(n+3)/2$. Thus the rank of the product matrix for $W(n)$ is at most $n2^{n-1} - n^2 + n(n+3)/2 = n2^{n-1} - n(n-3)/2$. For $n \geq 5$ this is strictly less than $n2^{n-1}$.

When $n = 3$ the bound above does not show singularity since it equals the size of the product matrix. In fact it is easy to show (by the row operations above) that in this case the matrix is nonsingular. This yields the result. □

3.3. $S(n)$ and $\widetilde{S}(n)$. Proposition 3.1 does not apply here. The highest degree occurring is $n - 2$ and yet if $r + s = n - 2$ then $\dim S_r = (s+1)\binom{n+1}{r+1}$ and $\dim S_s = (r+1)\binom{n+1}{s+1}$, these two dimensions being unequal in general. Thus even to obtain a first decomposition we need to work harder.

We shall not need the full details of how to multiply odd basis elements, but we now treat the necessary cases.

Using equation (2) we see that for the product of odd type I elements we have

$$[v_I \partial_i, v_J \partial_j] = \begin{cases} \pm v_{(I \cup J) \setminus \{i\}} \partial_j, & i \in J \text{ and } j \notin I \\ \pm v_{(I \cup J) \setminus \{j\}} \partial_i, & j \in I \text{ and } i \notin J \\ \pm v_{(I \cup J) \setminus \{i,j\}} h_{ij}, & i \in J \text{ and } j \in I \\ 0, & i \notin J \text{ and } j \notin I. \end{cases} \tag{7}$$

The third case follows from the following computation, where $I' = I \setminus \{j\}$, $J' = J \setminus \{i\}$.

$$\begin{aligned}
[v_I\partial_i, v_J\partial_j] &= (-1)^{1+p(J,i)}v_Iv_{J'}\partial_j + (-1)^{1+p(I,j)}v_Jv_{I'}\partial_i \\
&= (-1)^{1+p(J,i)+|I|-p(I,j)}v_{I'}v_jv_{J'}\partial_j \\
&\quad +(-1)^{1+p(I,j)+|J|-p(J,i)}v_{J'}v_iv_{I'}\partial_i \\
&= (-1)^{1+p(J,i)+|I|-p(I,j)+|J|-1}v_{I'}v_{J'}v_j\partial_j \\
&\quad +(-1)^{1+p(I,j)+|J|-p(J,i)+|I|-1}v_{J'}v_{I'}v_i\partial_i \\
&= (-1)^{p(I,j)+|J|-p(J,i)+|I|}\left[v_{I'}v_{J'}v_j\partial_j - v_{I'}v_{J'}v_i\partial_i\right] \\
&= (-1)^{p(I,j)+p(J,i)+1}v_{I'}v_{J'}h_{ij}
\end{aligned}$$

The last two equalities use the fact that $|I|$ and $|J|$ are even.

The product of a type I and a type II element leads to several cases, which can be summarized below.

(8)

$$[v_I\partial_i, v_Bh_{kl}] = \begin{cases} 0, & i \notin B \cup \{k,l\}, |\{k,l\} \cap I| \in \{0,2\} \\ \pm v_Iv_B\partial_i, & i \notin B \cup \{k,l\}, |\{k,l\} \cap I| = 1 \\ \pm v_Iv_{B\setminus\{i\}}h_{kl}, & i \in B, |\{k,l\} \cap I| \in \{0,2\} \\ \pm v_Iv_{B\setminus\{i\}}h_{ij}, & i \in B, j \notin B, \{i,j\} = \{k,l\} \\ \pm v_Iv_B\partial_i, & \{i,j\} = \{k,l\}, j \notin I \\ \pm 2v_Iv_B\partial_i, & \{i,j\} = \{k,l\}, j \in I \end{cases}$$

The computations are straightforward, using the fact that I is even and $|B|$ is odd. We give details for the 4th and 6th cases. For the first of these, suppose without loss of generality that $k \in I$ and $l \notin I$. Then

$$\begin{aligned}
[v_I\partial_i, v_Bh_{kl}] &= (-1)^{1+p(B,i)}v_Iv_{B\setminus\{i\}}h_{kl} + v_Bv_I\partial_i \\
&= (-1)^{1+p(B,i)}v_Iv_{B\setminus\{i\}}(-v_l\partial_l) + (-1)^{|B|-p(B,i)}v_{B\setminus\{i\}}v_iv_I\partial_i \\
&= (-1)^{1+p(B,i)}v_Iv_{B\setminus\{i\}}[v_i\partial_i - v_l\partial_l] \\
&= (-1)^{1+p(B,i)}v_Iv_{B\setminus\{i\}}h_{il}.
\end{aligned}$$

For the second suppose without loss of generality that $i = k$ and $l \in I$. Then

$$\begin{aligned}
[v_I\partial_i, v_Bh_{kl}] &= [v_I\partial_i, v_B(v_i\partial_i - v_l\partial_l)] \\
&= (-1)^{|B|}v_Iv_B\partial_i - v_Bv_I\partial_i \\
&= -2v_Iv_B\partial_i.
\end{aligned}$$

For a pair of type II elements we note that $h_{ij}(v_B)$ is either $\pm v_B$ if precisely one of i, j belongs to B, and zero otherwise. Since $[h_{ij}, h_{kl}] = 0$ the product $[v_Ah_{ij}, v_Bh_{kl}]$ is equal to $v_Ah_{ij}(v_B)h_{kl} + v_Bh_{kl}(v_B)h_{ij}$ and hence lies in the span of $v_A v_B h_{ij}$ and $v_A v_B h_{kl}$. Thus the subspace $S^{(2)} = \sum_r S_r^{(2)}$ is a subalgebra.

The multiplication in $\widetilde{S}(n)$ differs little from that in $S(n)$. If $v_J\partial_j$ has degree at least 1 then $|J \cap N| \geq 2$ and so $[(1+v_N)\partial_i, v_J\partial_j] = [\partial_i, v_J\partial_j]$, while $[(1+v_N)\partial_i, (1+v_N)\partial_j] = \partial_i(v_N)\partial_j + \partial_j(v_N)\partial_i$. Thus the only difference in a product matrix for $S(n)$ and one for $\widetilde{S}(n)$ is in the submatrix formed by the products of elements of degree -1.

3.3.1. *n even.*

Theorem 3.6. *Let $n \geq 4$ be even. Then $S(n)$ and $\widetilde{S}(n)$ satisfy Bell's criterion.*

Proof. Write $n = 2m$. We first treat the case of $S(2m)$ as the other case follows quickly from this. We make the specialization which sets all even type II variables to zero.

By the above the product of 2 type II elements is in the span of type II elements and hence specializes to zero.

Type I elements (I, i) and (J, j) are linked if and only if if and only if $|I| + |J| \leq 2m$, $I \cap J = \emptyset$ and either $i \in J$ or $j \in I$.

The product of a type I element $v_I\partial_i$ and a type II element $v_B h_{kl}$ remains nonzero only if $|I| + |B| \leq 2m - 1$ and $I \cap B = \emptyset$.

We obtain a (nonobvious) block decomposition of a product matrix M as follows. For each a with $0 \leq a \leq m-1$ define M_a to be the span of all variables of type (I,$2a-1$) and all variables of type (II,$2a+1$). It follows from the linking conditions above that $[M_a, M_b] = 0$ unless $a + b \leq m - 1$. Thus M has the reverse block upper triangular form

$$\begin{bmatrix} M_{0,0} & M_{0,1} & \cdots & M_{0,m-2} & M_{0,m-1} \\ M_{1,0} & M_{1,1} & \cdots & M_{1,m-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ M_{m-1,0} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Furthermore the blocks M_{ab} on the reverse diagonal, i.e. the ones with $a + b = m - 1$, are square:

$$\begin{aligned}\dim M_a &= \binom{2m}{2a}(2m-2a)+\binom{2m}{2a+1}(2m-2a-2)\\ &= \binom{2m}{2a}(2b+2)+\binom{2m}{2a+1}(2b)\\ &= \frac{(2m)!}{(2a+1)!(2b+1)!}(2b+1+2a)\\ &= \binom{2m}{2b}(2a+2)+\binom{2m}{2b+1}(2a)\\ &= \binom{2m}{2b}(2m-2b)+\binom{2m}{2b+1}(2m-2b-2)\\ &= \dim M_b.\end{aligned}$$

Thus it suffices to show that all such blocks M_{ab} with $a+b=m-1$ are nonsingular. Note that the $(-1,-1)$ product submatrix does not occur on the reverse diagonal and so we will dispose of both S and $\widetilde{S}$ with the same argument.

Now fix such a a and b and consider the block M_{ab}. Clearly M_{ab} has a 2×2 block form $\left(\begin{smallmatrix} X & Y\\ Z & 0\end{smallmatrix}\right)$ corresponding to the division of M_a and M_b into type I and type II variables.

We shall compute the rank of M_{ab} in stages. First note that in Y and Z the only nonzero entries occur when the last case of equation (8) holds. This is because $|B|+|I|=2m-1$ and B and I are disjoint, so one of k,l must belong to I. In fact if we define $z_i=v_{N\setminus\{i\}}\partial_i$ for $1\leq i\leq 2m$ then the only nonzero entries in Y and Z have the form $\pm 2z_i$ for some i. Furthermore the linking conditions can be expressed as follows. Choose a subset A of N of size $2k+1$. Then $(A\setminus\{i\},i)$ is linked to $(B,\{k,l\})$ if and only if $A\cap B=\emptyset$, $A\cup B=N$ and $i\in\{k,l\}$.

We first make the further specialization of all $z_i\mapsto 1/2$. Then the nonzero entries of Y and Z are all ± 1. Note that the variables occurring in X are all algebraically independent of the z_i and so X remains unchanged by this specialization. Furthermore the linking conditions are unchanged.

From now on we work modulo 2, i.e. we apply the natural homomorphism $\mathbb{Z}[L_+]\to\mathbb{Z}_2[L_+]$ to M. It suffices to show that the resulting matrix is nonsingular. In order to avoid excessive notation we use X,Y,Z to denote their images under this and all subsequent specializations.

First we compute the rank of Y. The linking conditions above show that the rows may be indexed by subsets A of N of size $2a+1$. The rows corresponding to a given A have nonzero entries only in columns indexed by $(B, \{i, j\})$ where $A \dot{\cup} B = N$ and $i, j \in A$. Thus the set of rows indexed by a given A_0 is independent of all rows indexed by all other subsets A since their nonzero entries occur in disjoint sets of columns. It therefore suffices to compute the rank of each submatrix Y_A formed by all rows corresponding to a given A.

Fix such a subset A. Now for a given row $(A \setminus \{i\}, i)$, there are two possibilities. If i is not the first element of A then there is a nonzero entry in column $(B, \{k, l\})$ if and only if $i = l$, so there is precisely one nonzero entry in this row. If i is the first element of A then there are nonzero entries in all columns $(B, \{i, l\})$. Thus by reordering columns we can bring Y_A to the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Adding all rows except the last to the last row we convert Y_A to $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$. It follows that the rank of Y_A is equal to $|A| - 1 = 2a$ and hence that the rank of Y equals $\binom{2m}{2a+1}(2a)$.

In fact the row operations above convert M to the form

$$\begin{bmatrix} Q & I \\ R & 0 \\ Z & 0 \end{bmatrix}.$$

Here R is the $\binom{2m}{2a+1} \times \binom{2m}{2a+1}(2b + 1)$ matrix consisting of all rows $(A \setminus \{i\}, i)$ with i the first element of A. Appropriate column operations then yield

$$\begin{bmatrix} 0 & I \\ R & 0 \\ Z & 0 \end{bmatrix}.$$

Now the rank of Z is $\binom{2m}{2b+1}(2b)$ by the same computation as for Y. In this case the analogous column operations followed by row operations convert M to

$$\begin{bmatrix} 0 & 0 & I \\ 0 & R' & 0 \\ I & 0 & 0 \end{bmatrix}.$$

Here R' is a $\binom{2m}{2a+1} \times \binom{2m}{2b+1}$ matrix which is therefore square since $2a + 1 + 2b + 1 = 2m$. Its rows are indexed by subsets A of N of size

$2a+1$ and its columns by subsets B of N of the complementary size $2b+1$.

It remains to compute the rank of R', and first we need to compute the entries of R'. The entry R'_{AB} with row index A and column index B is in the row indexed by $(A \setminus \{i_1\}, i_1)$ and column indexed by $(B \setminus \{j_1\}, j_1)$ where i_1, j_1 are respectively the first elements of A and B. It can be computed by applying the above row and column operations to the submatrix S_{AB} formed by all the row indices $(A \setminus \{i\}, i)$ and column indices $(B \setminus \{j\}, j)$. The entry R'_{AB} is in fact the sum of all nonzero entries in S_{AB}. We now compute this submatrix S_{AB}.

Now $(A \setminus \{i\}, i)$ links to $(B \setminus \{j\}, j)$ if and only if $A \cap B = \{i\}$ or $A \cap B = \{j\}$, the two possibilities being exclusive. Thus if $|A \cap B| \neq 1$ then $S_{AB} = 0$ and so $R'_{AB} = 0$.

One the other hand, if $|A \cap B| = 1$, let $\alpha \in A \cap B$. Then the linking conditions show that $(A \setminus \{i\}, i)$ and $(B \setminus \{j\}, j)$ are linked if and only if precisely one of i and j equals α, and the corresponding entry in M_{ab} is $\pm v_{A \cup B \setminus \{\alpha,\beta\}}$ where β is the one of i and j not equal to α. Thus R'_{AB} has the form $x_{A\Delta B} = \sum_{\beta \in A\Delta B} v_{A\Delta B \setminus \{\beta\}} \partial_\beta$.

In fact the distinct variables $x_{A\Delta B}$ are algebraically independent over K. To see this, note that since the $v_I \partial_i$ are all algebraically independent it suffices to show that the set $A\Delta B$ is determined by any one of the pairs $(A\Delta B \setminus \{\beta\}, \beta)$, and this is obvious.

The conditions of Lemma 3.2 apply with a slight change in notation and the result follows. □

3.3.2. *n odd.* Case (1)(i) of Proposition 3.1 applies since $\dim S_{-1} = n < n(n+1)/2 = \dim S_{n-2}$, and so Bell's criterion is not satisfied here. Here the zero pattern of the matrix was sufficient to make it singular. As noted in [Bel90], the same is true of the classical algebras $b(n)$. Since $S(3) \cong b(3)$ perhaps this is not surprising.

3.4. $H(n)$ **and** $\widetilde{H}(n)$**.** It is known that the multiplication in H satisfies

$$[D_f, D_g] = \pm D_{\{f,g\}}$$

where $\{f, g\} = \sum_i \partial_i(f)\partial_i(g)$. Note that this differs slightly from the notation in [Kac77], and that the exact multiplication formula is not needed for our purposes.

It follows from (1) above that $\partial_i(v_I)\partial_i(v_J) = 0$ unless $I \cap J = \{i\}$, whence

$$[x_I, x_J] = \begin{cases} \pm x_{I\Delta J}, & \text{if } |I \cap J| = 1, \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

where Δ denotes the symmetric difference (Boolean sum). Note that this implies that for a given $A, I \subseteq N$, the equation $[x_I, x_J] = \pm x_A$ has at most one solution for J. Furthermore this solution exists precisely when $I \not\subseteq A$ and $A \not\subseteq I$.

3.4.1. *n even.*

Theorem 3.7. *Let $n \geq 4$ be an even integer. Then $H(n)$ and $\widetilde{H}(n)$ satisfy Bell's criterion.*

Proof. Since $[\widetilde{H}, \widetilde{H}] = H = [H, H]$, it follows that the product matrices for both H and $\widetilde{H}$ may be assumed to be the same. Case (1)(i) of Proposition 3.1 applies since the highest degree occurring in $H(n)$ is $n-3$ and

$$\dim H_r = \binom{n}{r+2} = \binom{n}{s+2} = \dim H_s \qquad \text{if } r + s = n - 4.$$

Fix such a reverse diagonal block $H_{r,s}$. The conditions of Lemma 3.2 are satisfied, with the same notation, and the result follows. □

3.4.2. *n odd.* This case reduces rather easily to the previous one.

Theorem 3.8. *Let $n \geq 5$ be odd. Then $H(n)$ and $\widetilde{H}(n)$ satisfy Bell's criterion.*

Proof. Let $M, \widetilde{M}$ be the product matrices for $H(n), \widetilde{H}(n)$ respectively. The top degree $n-2$ occurring in $\widetilde{H}(n)$ is odd, so that $\widetilde{M}$ is obtained from M by adding another row and column. Since this procedure either leaves the rank unchanged or increases the rank by 1, it suffices to show that $\widetilde{M}$ is nonsingular.

We decompose $\widetilde{M}$ into 4 blocks as follows. Group the rows indexed by those I for which $n \in I$ together and follow them by the rows for which $n \notin I$. Do the same for the columns. This gives an obvious 2×2 block structure. Make the specialization which sets all even x_I with $n \in I$ to zero. Then $\widetilde{M}$ specializes to a matrix of the form $\left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$. It suffices to show that X and Y are nonsingular.

Now Y has entries which are the pairwise products of the x_I with $I \subseteq \{1, \ldots, n-1\}$ and hence is just a product matrix for $H(n-1)$. Thus Y is nonsingular by Theorem 3.7. Choose I with $n \in I$. Since $I \not\subseteq N \setminus \{n\}$ there is precisely one J with $n \in J$ for which $[x_I, x_J] = \pm x_{N \setminus \{n\}}$. Thus in X every row and column has precisely one occurrence of $\pm x_{N \setminus \{n\}}$, so specializing to zero all variables except this one yields a nonsingular monomial matrix. □

4. Comments

The author has written and used Maple code which generates the product matrices for the Cartan type algebras. It can be accessed at `http://www.math.auckland.ac.nz/~wilson/bellcrit.html`.

A natural question is: for the simple Lie superalgebras which do not satisfy Bell's criterion, which have (semi)prime enveloping algebras? It was shown in [KK96] that $U(b(n))$ is not semiprime but it has a unique minimal prime ideal. A similar argument (see [WP]) shows that $U(S(n))$ is not semiprime. However the case of $W(n)$ for odd $n \geq 5$ seems much harder.

The following is a list of open questions related to the the subject of this paper:

- If $U(L)$ is prime, and L is finite-dimensional, must L satisfy Bell's criterion?
- Is $U(W(n))$ prime for odd $n \geq 5$?
- Is $U(L)$ graded prime if and only if it is prime?
- Is $U(L)$ prime if and only if it is semiprime?
- If L is finite-dimensional, does $U(L)$ always have a unique minimal prime ideal?
- Does $U(S(n))$ have a unique minimal prime ideal?

References

[Bel90] Allen D. Bell, *A criterion for primeness of enveloping algebras of Lie superalgebras*, J. Pure Appl. Algebra **69** (1990), 111–120.

[Kac77] V. G. Kac, *Lie superalgebras*, Adv. in Math. **26** (1977), 8–96.

[KK96] Ellen Kirkman and James Kuzmanovich, *Minimal prime ideals in enveloping algebras of Lie superalgebras*, Proc. Amer. Math. Soc. **124** (1996), 1693–1702.

[Sch79] M. Scheunert, *The theory of Lie superalgebras*, Lecture Notes in Mathematics, vol. 716, Springer Verlag, 1979.

[Wil] Mark C. Wilson, *Primeness of the enveloping algebra of Hamiltonian superalgebras*, submitted.

[Wil96] Mark C. Wilson, *Primeness of the enveloping algebra of a Cartan type Lie superalgebra*, Proc. Amer. Math. Soc. **124** (1996), 383–387.

[WP] Mark C. Wilson and Geoffrey Pritchard, *Bell's criterion for $S(2n)$*, in preparation.

[WPW] Mark C. Wilson, Geoffrey Pritchard and David H. Wood, *Bell's primeness criterion for $W(2n+1)$*, Experiment. Math. (to appear).

Department of Mathematics, University of Auckland, PO Box 92019, Auckland, NEW ZEALAND