PRIMENESS OF THE ENVELOPING ALGEBRA OF THE SPECIAL LIE SUPERALGEBRAS

MARK C. WILSON AND GEOFFREY PRITCHARD

ABSTRACT. A primeness criterion due to Bell is shown to apply to the universal enveloping algebra of the Cartan type Lie superalgebras $S(V)$ and $\widetilde{S}(V;t)$ when $\dim V$ is even. This together with other recent papers yields

Theorem. *Let L be a finite-dimensional simple Lie superalgebra over an algebraically closed field of characteristic zero. Then L satisfies Bell's criterion* (so that $U(L)$ is prime), unless L is of one of the types: $b(n)$ for $n \geq 3$; $W(n)$ for odd $n \geq 5$; $S(n)$ for odd $n \geq 3$. □

On the other hand, if $\dim V$ is odd then $U(S(V))$ is never semiprime.

1. Introduction

It is well known that the universal enveloping algebra of a Lie algebra is always a domain and hence a prime ring, and that the analogous result is false for Lie superalgebras. In fact when L is a Lie superalgebra, necessary and sufficient conditions for the primeness of $U(L)$ are not known.

The results in this paper and its relatives cited below can be viewed in two ways: as answering this question for certain classes of simple superalgebras, or as a necessary first step toward the representation theorists' goal of a description of the prime and primitive spectra of $U(L)$.

For all background on Lie superalgebras we recommend [Sch79].

Let $L = L_+ \dotplus L_-$ be a finite-dimensional Lie superalgebra over a field K of characteristic zero, and let $U(L)$ be its universal associative enveloping (super)algebra. In [Bel90] Bell gave the following simple criterion for primeness of $U(L)$. Let $\{f_1, \ldots, f_n\}$ be a basis for the odd

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part L_- of L. Form the *product matrix* $M = ([f_i, f_j])$, considered as a matrix over the symmetric algebra $S(L_+)$. If $\det M \neq 0$ then $U(L)$ is prime.

The finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero were classified in [Kac77]. There are two essentially different kinds of such algebras, namely the *classical* algebras and those of *Cartan type*. The primeness question for the classical simple Lie superalgebras was settled in [Bel90] and [KK96]. The first author started the investigation of the Cartan type simple Lie superalgebras in [Wil96], and this was continued in [WPW] and concluded in this paper and [Wil].

In this paper we treat the case of the algebras which, in the notation of [Sch79], belong to the families $S, \widetilde{S}$. For lack of a standard name we call them the *special* Cartan type algebras.

Let $n \geq 3$ and let V be a finite-dimensional K-space. The Lie superalgebras $S(V)$ and $\widetilde{S}(V;t)$, as defined below, are simple and of Cartan type.

In this situation the following result holds (Theorem 3.4 and remarks in Section 3.1):

Theorem. *$S(V)$ satisfies Bell's criterion if and only if $\dim V$ is even, in which case $\widetilde{S}(V;t)$ also satisfies Bell's criterion.*

Though the failure of Bell's criterion to apply does not, as far as we know, rule out primeness, we also prove (Theorem 3.4 and Corollary 3.3):

Theorem. *$U(S(V))$ is prime if and only if $\dim V$ is even, in which case $U(\widetilde{S}(V;t))$ is also prime. If $\dim V$ is odd then $U(S(V))$ is not semiprime.* □

The results of this and the above-mentioned papers yield the following theorem.

Theorem. *Let L be a finite-dimensional simple Lie superalgebra over an algebraically closed field of characteristic zero. Then L satisfies Bell's criterion if and only if L is not of one of the types: $b(n)$ for $n \geq 3$; $W(n)$ for odd $n \geq 5$; $S(n)$ for odd $n \geq 3$.* □

Section 2 contains background on exterior algebras and Cartan type Lie superalgebras, and should be read, in order to fix notation, even by those familiar with such topics. The actual computations are in Section 3.

2. The special Lie superalgebras

2.1. Preliminaries. Again let K be a field of characteristic zero, n a positive integer and V an n-dimensional K-vector space. Let $\Lambda = \Lambda(V)$ be the Grassmann algebra of V. Recall that $\Lambda = \bigoplus_{r=0}^{n} \Lambda_r$ is an associative $\mathbb{Z}$-graded superalgebra. Fix a basis $\{v_1, \ldots, v_n\}$ for V. For each ordered subset $I = \{i_1, i_2, \ldots, i_r\}$ of $N = \{1, 2, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_r$, let v_I be the product $v_{i_1} v_{i_2} \cdots v_{i_r}$. The set of all such v_I forms a basis for Λ, where we interpret $1 = v_\emptyset$ as the empty product. We shall write simply v_i if $I = \{i\}$. The homogeneous component Λ_r is spanned by those v_I with $|I| = r$. The anticommutativity of multiplication in Λ implies that

$$v_I v_J = \begin{cases} \pm v_{I \cup J}, & \text{if } I \cap J = \emptyset, \\ 0, & \text{if } I \cap J \neq \emptyset. \end{cases} \tag{1}$$

The algebra $W = W(V)$ is the $\mathbb{Z}$-graded Lie superalgebra consisting of all superderivations of Λ. Every element of W maps V into Λ and since it is a superderivation it is completely determined by its action on the generating subspace V. It follows that W can be identified with $\Lambda \otimes_K V^*$ as a vector space. Under this identification the map $\partial_i = \partial/\partial_{v_i}$ corresponds to the dual of v_i, and every element of W can be uniquely written as $\sum_i \lambda_i \partial_i$ where $\lambda_i \in \Lambda$.

The set of all $v_I \partial_i$ is then a homogeneous basis for W, the degree of such an element being equal to $|I| - 1$. In degree zero we have $W_0 \cong V \otimes V^* \cong gl(n)$, the element $v_i \partial_j$ corresponding to the matrix unit e_{ij}.

The multiplication formula for *odd* elements is

$$[v_I \partial_i, v_J \partial_j] = v_I \partial_i(v_J) \partial_j + v_J \partial_j(v_I) \partial_i. \tag{2}$$

Note that this formula implies that the product on the left is zero if $|I \cap J| \geq 2$ or if both $i \notin J$ and $j \notin I$.

2.2. $S(V)$ and $\widetilde{S}(V; t)$. The subalgebra $S = S(V)$ of $W(V)$ consisting of all $\sum_i \lambda_i \partial_i$ such that $\sum_i \partial_i(\lambda_i) = 0$ is a Cartan type simple Lie superalgebra. It inherits a $\mathbb{Z}$-grading

$$S = \bigoplus_{r=-1}^{n-2} S_r$$

which is *consistent*, i.e. it induces the $\mathbb{Z}_2$-grading. If $V = K^n$ with standard basis $v_1, \ldots, v_n$ then we write $S(V) = S(n)$, and clearly $S(V) \cong S(n)$ for each V of dimension n. Thus it suffices to work with $S(n)$ and we shall henceforth do so.

We now describe the basis elements for $S(n)$ which we shall use for all computations in Section 3. A spanning set for each S_r is as follows and contains 2 distinct types of elements. Those elements of type (I,r) are all those of the form $v_I\partial_i$ with $i \notin I$ and $|I| = r+1$. Those of type (II,r) are of the form $v_A h_{ij}$ where $i, j \notin A$ and $|A| = r$. Here by definition $h_{ij} = v_i\partial_i - v_j\partial_j$.

The type I elements are all linearly independent, and their span $S_r^{(1)}$ is independent of the span $S_r^{(2)}$ of the type II elements. The type II elements are not independent however, since $h_{ij} + h_{jk} = h_{ik}$. We reduce the set of type II elements to a basis for $S_r^{(2)}$ as follows. For each A with $|A| = r$, order the complement $B = N \setminus A$ in the natural way as a subset of N and let i be the first element of B. Select those elements of the form $v_A h_{ij}$ where $i < j \in B$. These are easily seen to be independent and span $S_r^{(2)}$.

The calculations used to justify the last statement are essentially the same as those showing that the standard basis for the Lie algebra $sl(n)$ is indeed a basis. This is not an accident as the restriction of the isomorphism $W_0 \cong gl(n)$ carries S_0 onto $sl(n)$. Under this the type (I,0) basis elements $v_i\partial_j$ correspond to the off-diagonal matrix units e_{ij} and the type (II,0) basis elements h_{1j} to the diagonal elements $e_{11} - e_{jj}$.

We shall need the dimension formulas

$$\begin{aligned} \dim S_r^{(1)} &= (n-r-1)\binom{n}{r+1} \\ \dim S_r^{(2)} &= (n-r-1)\binom{n}{r}. \end{aligned}$$

We shall not need the full details of how to multiply the basis elements above. The formulas below are readily obtained using (1) and the fact that the ∂_i are odd superderivations. Thus if $p(B,i)$ denotes the position of i in the ordered set B, we have $\partial_i(v_B) = (-1)^{1+p(B,i)} v_{B\setminus\{i\}}$ if $i \in B$, and $\partial_i(v_B) = 0$ otherwise. Also $h_{ij}(v_B)$ is either $\pm v_B$ if precisely one of i, j belongs to B, and zero otherwise.

Equation (2) implies that for a product of two odd type I elements we have

$$[v_I\partial_i, v_J\partial_j] = \begin{cases} \pm v_I v_{J\setminus\{i\}}\partial_j, & i \in J \text{ and } j \notin I \\ \pm v_J v_{I\setminus\{j\}}\partial_i, & j \in I \text{ and } i \notin J \\ \pm v_{I\setminus\{j\}} v_{J\setminus\{i\}} h_{ij}, & i \in J \text{ and } j \in I \\ 0, & i \notin J \text{ and } j \notin I. \end{cases} \tag{3}$$

The 3rd case follows from the following computation, where $I' = I \setminus \{j\}, J' = J \setminus \{i\}$.

$$\begin{aligned}
[v_I\partial_i, v_J\partial_j] &= (-1)^{1+p(J,i)}v_Iv_{J'}\partial_j + (-1)^{1+p(I,j)}v_Jv_{I'}\partial_i \\
&= (-1)^{1+p(J,i)+|I|-p(I,j)}v_{I'}v_jv_{J'}\partial_j \\
&\quad +(-1)^{1+p(I,j)+|J|-p(J,i)}v_{J'}v_iv_{I'}\partial_i \\
&= (-1)^{1+p(J,i)+|I|-p(I,j)+|J|-1}v_{I'}v_{J'}v_j\partial_j \\
&\quad +(-1)^{1+p(I,j)+|J|-p(J,i)+|I|-1}v_{J'}v_{I'}v_i\partial_i \\
&= (-1)^{p(I,j)+|J|-p(J,i)+|I|}\left[v_{I'}v_{J'}v_j\partial_j - v_{I'}v_{J'}v_i\partial_i\right] \\
&= (-1)^{p(I,j)+p(J,i)+1}v_{I'}v_{J'}h_{ij}.
\end{aligned}$$

The last two equalities use the fact that $|I|$ and $|J|$ are even.

The product of an odd type I and an odd type II element leads to the following mutually exclusive and exhaustive cases.

(4)

$$[v_I\partial_i, v_Bh_{kl}] = \begin{cases} 0, & i \notin B \cup \{k,l\}, |\{k,l\} \cap I| \in \{0,2\} \\ \pm v_Iv_B\partial_i, & i \notin B \cup \{k,l\}, |\{k,l\} \cap I| = 1 \\ \pm v_Iv_{B\setminus\{i\}}h_{kl}, & i \in B, |\{k,l\} \cap I| \in \{0,2\} \\ \pm v_Iv_{B\setminus\{i\}}h_{ij}, & i \in B, \{k,l\} \cap I = \{j\} \\ \pm v_Iv_B\partial_i, & \{i,j\} = \{k,l\}, j \notin I \\ \pm 2v_Iv_B\partial_i, & \{i,j\} = \{k,l\}, j \in I \end{cases}$$

The computations are straightforward, using the fact that $|I|$ is even and $|B|$ is odd. We give details for the 4th and 6th cases. For the first of these, suppose without loss of generality that $k \in I$ and $l \notin I$. Then

$$\begin{aligned}
[v_I\partial_i, v_Bh_{kl}] &= (-1)^{1+p(B,i)}v_Iv_{B\setminus\{i\}}h_{kl} + v_Bv_I\partial_i \\
&= (-1)^{1+p(B,i)}v_Iv_{B\setminus\{i\}}(-v_l\partial_l) + (-1)^{|B|-p(B,i)}v_{B\setminus\{i\}}v_iv_I\partial_i \\
&= (-1)^{1+p(B,i)}v_Iv_{B\setminus\{i\}}[v_i\partial_i - v_l\partial_l] \\
&= (-1)^{1+p(B,i)}v_Iv_{B\setminus\{i\}}h_{il}.
\end{aligned}$$

For the second suppose without loss of generality that $i = k$ and $l \in I$. Then

$$\begin{aligned}
[v_I\partial_i, v_Bh_{kl}] &= [v_I\partial_i, v_B(v_i\partial_i - v_l\partial_l)] \\
&= (-1)^{|B|}v_Iv_B\partial_i - v_Bv_I\partial_i \\
&= -2v_Iv_B\partial_i.
\end{aligned}$$

For a pair of odd type II elements we note that since $[h_{ij}, h_{kl}] = 0$ the product $[v_A h_{ij}, v_B h_{kl}]$ is equal to $v_A h_{ij}(v_B) h_{kl} + v_B h_{kl}(v_B) h_{ij}$ and hence lies in the span of $v_A v_B h_{ij}$ and $v_A v_B h_{kl}$. Thus the subspace $S^{(2)} = \sum_r S_r^{(2)}$ is a subalgebra.

The simple Lie superalgebra $\widetilde{S} = \widetilde{S}(V;t)$ is defined, only when $n = \dim V$ is even, as follows. Let t be a nonzero element of Λ_n. For each r with $0 \leq r \leq n-2$, $\widetilde{S}_r(V)$ is just $S_r(V)$. The difference is that the component $\widetilde{S}_{-1}$ has a basis consisting of all $(1+t)\partial_i$ with $i \in N$. Then $\widetilde{S}$ is the direct sum of the $\widetilde{S}_r$ but this is not an algebra grading. Note that if $V = K^n$ then necessarily t is a nonzero scalar multiple of v_N, and it is easily seen that $\widetilde{S}(V;t) \cong \widetilde{S}(K^n; v_N) := \widetilde{S}(n)$. Thus it suffices to work with $\widetilde{S}(n)$ and we shall henceforth do so.

The multiplication in $\widetilde{S}(n)$ differs little from that in $S(n)$. If $v_J \partial_j$ has degree at least 1 then $|J \cap N| \geq 2$ and so $[(1+v_N)\partial_i, v_J\partial_j] = [\partial_i, v_J \partial_j]$, while $[(1+v_N)\partial_i, (1+v_N)\partial_j] = \partial_i(v_N)\partial_j + \partial_j(v_N)\partial_i$. Thus the only difference in a product matrix for $S(n)$ and one for $\widetilde{S}(n)$ need be in the submatrix formed by the products of elements of degree -1.

3. Computation

3.1. $\dim V$ **odd.** It was shown in [Wil96] that if $\dim V$ is odd then $S(V)$ does not satisfy Bell's criterion. In fact a product matrix for $S(n)$ in this case has the form

$$\begin{pmatrix} S_{-1,-1} & S_{-1,1} & \dots & \dots & S_{-1,n-2} \\ S_{1,-1} & S_{1,1} & \dots & S_{1,n-4} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{n-2,-1} & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Here $S_{i,j}$ is the product submatrix formed by the products of elements from S_i with those from S_j. The block $S_{n-2,-1}$ is therefore of size $n(n+1)/2 \times n$. Thus it has more rows than columns, and it follows immediately that the product matrix is singular (in fact every term in the full expansion of the determinant is zero).

There still remains the question of whether $U(L)$ is prime when $L = S(n)$ and n is odd. In fact the above inequality of dimensions is enough to show that this is not the case.

Proposition 3.1. *Suppose L is a direct sum $A \dotplus B \dotplus C$ of subspaces where A and B are odd abelian, $[B, C] \subseteq B$ and $\dim A < \dim B$. Then $U(L)$ is not semiprime.*

Proof. Write $s = \dim A, t = \dim B$, $U = U(L)$. For V a subspace of L, by V^n we mean the n-th associative power of V inside U.

We claim that $A^iB^j \subseteq B^{j-i}U$ whenever $0 \leq i \leq j$. Now $AB^j \subseteq B^jA+[A,B^j] \subseteq \sum_{k+l=j-1} B^k[A,B]B^l$. Also $[A,B] \subseteq L_+ \subseteq C$. Since B is stable under $\operatorname{ad} C$, so is each B^l and so $CB^l \subseteq B^lC+B^l \subseteq B^lU$. Thus $AB^j \subseteq B^jA+ \subseteq B^{j-1}U \subseteq B^{j-1}U$. The claim follows immediately by induction on i.

We now claim that B^t generates a nilpotent ideal of U. It suffices to show that $UB^t \subseteq BU$ since $B^tB = 0$. Now U is the sum of all $A^iB^jC^k$ with $i,j,k \geq 0$, and $A^iB^jC^kB^t \subseteq A^iB^{j+t}C^k \subseteq B^{j+t-i}U$. Since $s < t$ and $A^i = 0$ whenever $i > s$, all $t-i$ occurring are at least 1 and so $B^{j+t-i}U \subseteq BU$.

□

Proposition 3.2. *Let L be a consistently $\mathbb{Z}$-graded Lie superalgebra such that*
(1) $L = \bigoplus_{r=-1}^{m} L_r$ *and m is odd*
(2) $\dim L_{-1} < \dim L_m$
Then $U(L)$ is not semiprime.

Proof. The hypotheses guarantee that the previous proposition applies with $A = L_{-1}, B = L_m$ and C the sum of the remaining L_i. □

Corollary 3.3. *If n is odd then $U(S(n))$ is not semiprime.* □

Proposition 3.1 applies also to the classical algebra $L = b(n)$, in which case it yields the same argument as in [KK96]. Since $b(3) \cong S(3)$ a common argument might be expected.

3.2. $\dim V$ **even.** We now consider the much more difficult case where n is even. The proof of the next result is rather involved but ultimately rests on little more than a judicious choice of specializations and row and column operations.

Theorem 3.4. *Let $n \geq 4$ be even. Then $S(n)$ and $\widetilde{S}(n)$ satisfy Bell's criterion.*

Proof. We first treat the case of $S(n)$ as the other case follows quickly from this. Write $n = 2m$.

We make the specialization which sets all even type II variables to zero. Let M denote the resulting specialization of the product matrix for $S(2m)$.

By the above the product of two type II elements is in the span of type II elements and hence specializes to zero.

By (3) the product of two type I elements $v_I\partial_i$ and $v_J\partial_j$ remains nonzero if and only if $|I| + |J| \leq 2m$, $I \cap J = \emptyset$ and precisely one of the conditions $i \in J$, $j \in I$ holds.

By (4) the product of a type I element $v_I\partial_i$ and a type II element $v_B h_{kl}$ remains nonzero only if $|I|+|B|\le 2m-1$ and $I\cap B=\emptyset$.

We obtain a (nonobvious) block decomposition of M as follows. For each a with $0\le a\le m-1$ define M_a to be the span of all variables of type (I,$2a-1$) and all variables of type (II,$2a+1$). It follows from above that $[M_a,M_b]=0$ unless $a+b\le m-1$. Thus M has the reverse block upper triangular form

$$\begin{bmatrix} M_{0,0} & M_{0,1} & \cdots & M_{0,m-2} & M_{0,m-1} \\ M_{1,0} & M_{1,1} & \cdots & M_{1,m-2} & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ M_{m-1,0} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Furthermore the blocks M_{ab} on the reverse diagonal, i.e. the ones with $a+b=m-1$, are square. To see this, we compute:

$$\begin{aligned} \dim M_a &= \binom{2m}{2a}(2m-2a)+\binom{2m}{2a+1}(2m-2a-2) \\ &= \binom{2m}{2a}(2b+2)+\binom{2m}{2b+1}(2b) \\ &= \frac{(2m)!}{(2a+1)!(2b+1)!}(2b+1+2a) \\ &= \binom{2m}{2b}(2a+2)+\binom{2m}{2b+1}(2a) \\ &= \binom{2m}{2b}(2m-2b)+\binom{2m}{2b+1}(2m-2b-2) \\ &= \dim M_b. \end{aligned}$$

Thus it suffices to show that all such blocks M_{ab} with $a+b=m-1$ are nonsingular.

Now fix such a a and b and consider the block M_{ab}. Clearly M_{ab} has a 2×2 block form $\left(\begin{smallmatrix} X & Y \\ Z & 0 \end{smallmatrix}\right)$ corresponding to the division of M_a and M_b into type I and type II variables.

We now derive conditions which are equivalent to the product of basis elements remaining nonzero under this specialization. We refer to these as linking conditions. This has an obvious graph-theoretic interpretation which we now describe. If in the symmetric matrix M_{ab} we replace every nonzero entry with a 1, we have the adjacency matrix of a unique graph $G(M_{ab})$. Thus the vertices of $G(M_{ab})$ are labelled by the basis elements of M_a and M_b and there is an edge joining two vertices if and only if the corresponding product gives a nonzero entry of M_{ab}.

First note that in Y and Z the only nonzero entries occur when the last case of equation (4) holds. This is because $|B| + |I| = 2m - 1$ and B and I are disjoint, so one of k, l must belong to I and the other must equal i. In fact if we define $z_i = v_{N\setminus\{i\}}\partial_i$ for $1 \leq i \leq 2m$ then the only nonzero entries in Y and Z have the form $\pm 2z_i$ for some i. Furthermore the linking conditions for Y (respectively Z) can be expressed as follows. Choose a subset A of N of size $2a+1$ (respectively $2b+1$). Then $(A\setminus\{i\}, i)$ is linked to $(B, \{k, l\})$ if and only if $A \cap B = \emptyset$, $A \cup B = N$ and $i \in \{k, l\}$.

We first make the further specialization of all $z_i \mapsto 1/2$. The resulting matrix is defined over the polynomial ring $\mathbb{Z}[L_+]$. In order to avoid excessive notation we use X, Y, Z to denote their images under this and all subsequent specializations. Then the nonzero entries of Y and Z are all ± 1. Note that the variables occurring in X are all algebraically independent of the z_i and so X remains unchanged by this specialization.

From now on we work modulo 2, i.e. we apply the natural homomorphism $\mathbb{Z}[L_+] \to \mathbb{Z}_2[L_+]$. It suffices to show that the resulting matrix is nonsingular.

First we compute the rank of Y. The linking conditions above show that the rows may be indexed by subsets A of N of size $2a + 1$. The rows corresponding to a given A have nonzero entries only in columns indexed by $(B, \{i, j\})$ where $B = N \setminus A$ and $i, j \in A$. Thus the set of rows indexed by a given A_0 is independent of all rows indexed by all other subsets A since their nonzero entries occur in disjoint sets of columns. It therefore suffices to compute the rank of each submatrix Y_A formed by all rows corresponding to a given A.

Fix such a subset A. Now for a given row $(A \setminus \{i\}, i)$, there are two possibilities. If i is not the first element of A then there is a nonzero entry in column $(B, \{k, l\})$ if and only if $i = l$, so there is precisely one nonzero entry in this row. If i is the first element of A then there are nonzero entries in all columns $(B, \{i, l\})$. Thus by reordering rows and columns we can bring Y_A to the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Adding all rows except the last to the last row we bring Y_A to the form $\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$. It follows that the rank of Y_A is equal to $|A| - 1 = 2a$ and hence that the rank of Y equals $\binom{2m}{2a+1}(2a)$.

In fact the row operations above convert M_{ab} to the form

$$\begin{bmatrix} Q & I \\ R & 0 \\ Z & 0 \end{bmatrix}.$$

Here R is a $\binom{2m}{2a+1} \times \binom{2m}{2a+1}(2b+1)$ matrix with rows indexed by all $(A\backslash\{i\}, i)$ with i the first element of A. Appropriate column operations then yield

$$\begin{bmatrix} 0 & I \\ R & 0 \\ Z & 0 \end{bmatrix}.$$

Now the rank of Z is $\binom{2m}{2b+1}(2b)$ by the same computation as for Y. In this case the analogous column operations followed by row operations convert M_{ab} to

$$\begin{bmatrix} 0 & 0 & I \\ 0 & R' & 0 \\ I & 0 & 0 \end{bmatrix}.$$

Here R' is a $\binom{2m}{2a+1} \times \binom{2m}{2b+1}$ matrix which is therefore square since $2a + 1 + 2b + 1 = 2m$. Its rows are indexed by subsets A of N of size $2a + 1$ and its columns by subsets B of N of the complementary size $2b + 1$.

It remains to show that R' is nonsingular, and first we need to compute the entries of R'. The entry R'_{AB} with row index A and column index B is in the row indexed by $(A \setminus \{i_1\}, i_1)$ and column indexed by $(B\setminus\{j_1\}, j_1)$ where i_1, j_1 are respectively the first elements of A and B. It can be computed by applying the above row and column operations to the submatrix S_{AB} formed by all the row indices $(A \setminus \{i\}, i)$ and column indices $(B \setminus \{j\}, j)$. In fact R'_{AB} is the sum of all entries in S_{AB}. We now compute S_{AB} and hence R'_{AB}.

Now $(A\backslash\{i\}, i)$ links to $(B\backslash\{j\}, j)$ in M_{ab} if and only if $A \cap B = \{i\}$ or $A \cap B = \{j\}$, the two possibilities being exclusive. Thus if $|A \cap B| \neq 1$ then the entries of S_{AB} are all zero and so $R'_{AB} = 0$.

One the other hand, if $|A \cap B| = 1$, let $\alpha \in A \cap B$. Then $(A \setminus \{i\}, i)$ and $(B \setminus \{j\}, j)$ are linked if and only if precisely one of i and j equals α, and the corresponding entry in M_{ab} is $\pm v_{A\cup B\backslash\{\alpha,\beta\}}$ where β is the one of i and j not equal to α. Thus R'_{AB} has the form

$$x_{A\Delta B} = \sum_{\beta \in A\Delta B} v_{A\Delta B \setminus \{\beta\}} \partial_\beta,$$

where Δ denotes the symmetric difference. We have therefore computed the matrix R'. In what follows we use further specializations to prove it nonsingular.

In fact the distinct $x_{A\Delta B}$ are algebraically independent over K. To see this, note that since the $v_I \partial_i$ are all algebraically independent it suffices to show that the set $A\Delta B$ is determined by any one of the pairs $(A\Delta B \setminus \{\beta\}, \beta)$, and this is obvious. Furthermore in each row or column a given variable $x_{A\Delta B}$ can appear at most once, since if either A or B is fixed, then $A\Delta B$ determines the other.

The sets $A\Delta B$ all have size $n-2$, so the variables $x_{A\Delta B}$ can be relabelled by the 2-element complements $\{\alpha, \beta\}$. Write $y_{\alpha,\beta} = x_{A\Delta B}$ in this case. We now specialize to zero all of these $y_{\alpha,\beta}$ except when $\{\alpha, \beta\}$ is one of the m sets $\{1,2\}, \{3,4\}, \ldots, \{2m-1, 2m\}$. This converts R' to a matrix R'' and it now suffices to show that R'' is nonsingular over the function field generated by the $y_{\alpha,\beta}$.

If the entry of R'' corresponding to row A and column B is $y_{\alpha,\beta}$, we say that A and B are linked in $G(R'')$ by an edge of colour $\{\alpha, \beta\}$. Note that by the above a given colour can occur at most once at each vertex.

We now obtain a further block decomposition of R'' by showing that the set of colours occurring at a given vertex is constant on each component. To this end, we first show that vertices distance 2 apart have the same colours. Suppose that A and B are linked by an edge of colour $\{\alpha, \beta\}$. Then without loss of generality $A \cap B = \{\alpha\}$ and $A \cup B = N \setminus \{\beta\}$. Let C be linked to B by an edge of a different colour $\{\gamma, \delta\}$ then $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$ and we may assume that $B \cap C = \{\gamma\}$ and $B \cup C = N \setminus \{\delta\}$. Then $\beta \in C$. Let $X = B \cup \{\beta, \delta\} \setminus \{\alpha, \gamma\}$. Then $|X| = |B|$, $C \cap X = \{\beta\}, C \cup X = N \setminus \{\alpha\}$ and so C and X are linked by an edge of colour $\{\alpha, \beta\}$. Thus every colour occurring at A also occurs at C, and by symmetry A and C have the same colours.

It follows that if A and B are adjacent vertices then they have the same colours, since if an edge of some colour joins A and D, then B and D have the same colours by above and so the given colour occurs at B. Thus the set of colours occurring at a vertex is constant on components.

Hence there is a direct sum decomposition of R'', each block being parametrized by a given set of colours. Fix such a set of colours which occurs and consider the block which it parametrizes. Then every variable occurring is present exactly once in each row and column. Hence specializing all but one of the variables to zero yields a nonsingular monomial matrix. This yields the result for $S(n)$.

The product matrix for $\widetilde{S}(n)$ is the same as that for $S(n)$ except for the $(-1,-1)$ submatrix. Since this is contained in the blocks M_{ab} which are off the reverse diagonal, the above argument carries over verbatim and yields the result for $\widetilde{S}(n)$. This completes the proof. □

4. Comments

Let L be a finite-dimensional Lie superalgebra in characteristic zero.

We do not know of an L such that $U(L)$ is semiprime but not prime. It is known that for L classical simple or nilpotent, $U(L)$ always has a unique minimal prime ideal (see [KK96]). If this were true for all L then $U(L)$ would always be prime if and only if it were semiprime, since a unique minimal prime in a noetherian ring is nilpotent. Clearly $U(S(n))$, for odd $n \geq 3$, should be investigated in this regard.

It is still unknown whether Bell's criterion is necessary, as well as sufficient, for $U(L)$ to be prime.

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Department of Mathematics, University of Auckland, Private Bag 92019 Auckland, New Zealand
E-mail address: wilson@math.auckland.ac.nz

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA
E-mail address: Geoffrey.Pritchard@math.tamu.edu