Asymptotics of the minimum manipulating coalition size for positional voting rules under impartial-culture behaviour

Geoffrey Pritchard* and Mark C. Wilson
University of Auckland

February 17, 2009

Abstract

We consider the problem of manipulation of elections using positional voting rules under Impartial Culture voter behaviour. We consider both the logical possibility of coalitional manipulation, and the number of voters that must be recruited to form a manipulating coalition. It is shown that the manipulation problem may be well approximated by a very simple linear program in two variables. This permits a comparative analysis of the asymptotic (large-population) manipulability of the various rules. It is seen that the manipulation resistance of positional rules with 5 or 6 (or more) candidates is quite different from the more commonly analyzed 3- and 4-candidate cases.

Key words and phrases: scoring rule, social choice, manipulation, Impartial Culture, Borda, plurality, anti-plurality, asymptotic, probability, linear programming.

JEL Classification Numbers: D71, D72.

1 Introduction

In 1973–75 Gibbard and Satterthwaite published a fundamental impossibility theorem which states that every non-dictatorial social choice function, whose range contains at least three alternatives, at certain profiles can be manipulated by a single individual voter (Gibbard (1973), Satterthwaite (1975)). After that, the natural question arose: if there are no perfect rules, which ones are the best, i.e. least manipulable? To this question there can be no absolute answer – it depends both on the behaviour of the voters, and on the measure used to quantify the term "manipulability".

Among models of voter behaviour, the following two have gained the most attention (Berg and Lepelley (1994), Kelly (1993), Saari (1990)). The Impartial Culture (IC) model assumes that voters are independent, and that each voter is equally likely to express any of the possible preference orders among the candidates. The Impartial Anonymous Culture (IAC) model assumes

*Corresponding author. Dept. of Statistics, University of Auckland, Private Bag 92019, Auckland, New Zealand; g.pritchard@auckland.ac.nz; +64 9 373 7599 ext. 87400

some degree of dependency among the voters. In the present paper, we consider the IC model. This is a very challenging model for social choice rules: since no candidate is inherently favoured by the culture, the voters' collective expressed opinions will create (by chance alone) only slight distinctions between candidates, and it is unlikely that there will be any clear winner. In particular, Condorcet's Paradox occurs more frequently in IC electorates than in more realistically distributed ones (Regenwetter et al (2006)). These features have led to criticism of the IC model as somewhat unrealistic (see Regenwetter et al (2006)). However, the same features make IC a useful setting in which to study manipulability, since manipulation becomes much easier when the margin of victory is narrow, or when the victor is not a Condorcet winner. Thus, we choose IC in order to focus on situations likely to be manipulable. A necessary caveat to this choice is that the more manipulable parts of another distribution of profiles might not, themselves, resemble the IC distribution.

A realistic study of manipulation would lead us into the theory of political coalitions as canvassed in Riker (1962) and (even more qualitatively) Riker (1986). Such an approach soon encompasses considerations (e.g. changing the nature of the issue being voted on) beyond the reach of the mathematical simplicities of social choice theory. Instead, quantitative studies to date have focused on rather stylized notions of manipulability. The most popular measure has been the probability that the votes fall in such a way as to create the (coalitional or individual) "logical possibility of manipulation". This means that some coalition of voters (or individual voter) with incentive to do so can change the election result by voting insincerely. Note in particular that counterthreats are not considered – the manipulator(s) are not opposed by the other, non-strategic voters – and so the existence of a possible manipulation does not imply its presence in a Nash equilibrium in the game-theoretic sense. This model of manipulability has been very widely studied (Baharad and Neeman (2002), Chamberlin (1985), Ju (2005), Kelly (1993), Kim and Roush (1996), Lepelley and Mbih (1987, 1994), Maus et al (2007), Nitzan (1985), Pritchard and Slinko (2006), Pritchard and Wilson (2007), Saari (1990)). For the case of individual manipulation, some elaborations (the number of individuals who may manipulate, their freedom to do so, and the benefit they derive therefrom) are studied in Aleskerov and Kurbanov (1998) and Smith (1999). The positional (scoring) voting rules have been particular favourites, and significant progress has been made in comparing them. In his seminal paper Saari (1990), Saari showed that in his "geometric" model, Borda's rule is the least manipulable for the three-alternative case in relation to individual manipulation, but that this does not extend to the case of four alternatives.

However, the mere possibility of manipulation sheds little light on the difficulty of carrying it out. For example, how might voters come to discover who can be persuaded to vote insincerely in order to effect a better outcome? Without going into detail concerning such a process, it is clear that the size of the required coalition is of central importance. Intuitively, a situation is more resistant to manipulation if many voters must be recruited to assemble the manipulating coalition, and less resistant if only a few voters are required. In this paper, we consider the probability that a coalition *of at most k voters* can manipulate ($k = 1, 2, \ldots$). Equivalently, we study the probability distribution of the size of the smallest manipulating coalition (a random variable). Similar ideas are explored, in a more limited way, in Pritchard and Slinko (2006) and Pritchard and Wilson (2007).

We use the following notation and assumptions throughout. An election is held to choose one from among m candidates ($m \geq 3$). There are n voters, who hold opinions according to the IC model. That is, each voter is (independently) of one of the $m!$ possible types (preference orders on the candidates), each type being equally likely. The election uses the positional voting rule with score vector $w = (w_1, \ldots, w_m)$, where $1 = w_1 \geq w_2 \geq \cdots \geq w_m = 0$. That is, a vote ranking candidate α in ith place contributes w_i to the score of α, and the candidate with the greatest total score is declared the winner. The possibility of a tie for first place will not be considered in this paper, as Proposition 3 makes it largely irrelevant; it is discussed in detail in Pritchard and Wilson (2007). We aim to describe the limiting probability distribution of the minimum manipulating coalition size as $n \to \infty$, and to use this as a criterion for comparing the rules.

The remainder of this paper is organized as follows. The next section records some basic results regarding IC behaviour in large populations. Section 3 is the theoretical core of the paper: it defines the manipulation problem as an integer linear program and then, through a series of simplifications, shows how this may be replaced by a much simpler linear program. The lengthier proofs in this section have been relegated to appendices; readers impatient to reach the main results may wish to skip them at a first reading. The desired limiting probability distributions are derived in section 4, and used in section 5 to compare the rules. Section 6 contains some conclusions.

2 Asymptotic results for large electorates of IC voters

Let C be the set of candidates, and T the set of all voter types (i.e. all permutations of C). Let $N = (N_t)_{t \in T}$ be the random vector giving the number of voters of each type (so $\sum_{t \in T} N_t = n$). This vector is sometimes termed a "voting situation". For IC voter behaviour, N has a multinomial probability distribution with mean $(n/m!)\mathbf{1}$. (Here and subsequently, the notation $\mathbf{1}$ is used to denote a vector whose entries are all 1.) Under the asymptotic conditions of interest to us, this may be approximated by a multivariate normal distribution.

Proposition 1.

$$\frac{N - np\mathbf{1}}{\sqrt{n}} \xrightarrow{D} N(0, \Sigma),$$

where $p = 1/m!$, and Σ is the $m!$-by-$m!$ matrix with entries

$$\Sigma_{st} = \begin{cases} p(1-p), & \text{if } s = t \\ -p^2, & \text{if } s \neq t \ . \end{cases}$$

Remark. Here and in the rest of this paper, the notation $\xrightarrow{D}$ denotes convergence in distribution (see Durrett (1996), Ch. 2). Note that the limiting multivariate normal distribution is degenerate (Σ is a singular matrix).

Proof. Since IC voters obtain their types at random, and independently, we have $N = \sum_{i=1}^n X_i$, where $X_1, \ldots, X_n$ are independent and have probability distribution assigning probability $1/m!$ to each of the unit vectors of $\mathbf{R}^T$. Note that $E[X_1] = p\mathbf{1}$ and the covariance matrix of X_1 is Σ. The result then follows by the central limit theorem (Durrett (1996), p.170). ∎

We use the notation $\sigma_t(\alpha)$ for the contribution to candidate α's score made by a vote of type t (so if t ranks α in ith place, then $\sigma_t(\alpha) = w_i$). The total score of α is then

$$|\alpha| = \sum_{t \in T} N_t \sigma_t(\alpha).$$

Let $S = (|\alpha|)_{\alpha \in C}$ be the vector of candidates' scores (the "scoreboard"). Proposition 1 immediately gives a central limit result for S, too.

Proposition 2.

$$\frac{S - n\bar{w}\mathbf{1}}{\sqrt{n}} \xrightarrow{D} \sigma_w \left(\frac{m}{m-1}\right)^{1/2} (Z - \bar{Z}\mathbf{1}),$$

where $\bar{w}$, σ_w are the mean and standard deviation of the score vector w (i.e. $\bar{w} = (w_1 + \cdots + w_m)/m$ and $\sigma_w^2 = (w_1^2 + \cdots + w_m^2)/m - (\bar{w})^2$); Z is a vector, indexed by C, of independent standard normal random variables; and $\bar{Z} = \frac{1}{m}\sum_\alpha Z_\alpha$.

Proof. Let $Y = (Y_t)_{t \in T} \sim N(0, \Sigma)$. From Proposition 1 we have

$$\frac{S - n\bar{w}\mathbf{1}}{\sqrt{n}} \xrightarrow{D} U,$$

where $U_\alpha = \sum_{t \in T} Y_t \sigma_t(\alpha)$. It only remains to show that U and $\sigma_w \left(\frac{m}{m-1}\right)^{1/2} (Z - \bar{Z}\mathbf{1})$ have the same multivariate normal distribution. For this, it suffices to observe that they have the same mean (zero), variances, and covariances. It is routine to check that

$$\mathrm{Var}\left(\sum_{t \in T} Y_t \sigma_t(\alpha)\right) = \sigma_w^2 = \sigma_w^2 \left(\frac{m}{m-1}\right) \mathrm{Var}\left(Z_\alpha - \bar{Z}\right)$$

for $\alpha \in C$, and for distinct $\alpha, \beta \in C$

$$\mathrm{Cov}\left(\sum_{t \in T} Y_t \sigma_t(\alpha), \sum_{t \in T} Y_t \sigma_t(\beta)\right) = \frac{-\sigma_w^2}{m-1} = \sigma_w^2 \left(\frac{m}{m-1}\right) \mathrm{Cov}\left(Z_\alpha - \bar{Z}, Z_\beta - \bar{Z}\right).$$

∎

Proposition 2 implies that under IC behaviour in large electorates, the average candidate's score will be of order n, but the variability among the scores will be of order only $\sqrt{n}$. Consequently, most elections will result in all candidates receiving relatively similar scores, and the margin of victory will be small. However, exact ties in the scores become increasingly rare as the number of voters increases. The following result establishes this formally.

Proposition 3.

$$P\,(\text{all candidates' scores are numerically distinct}) \to 1 \qquad \text{as } n \to \infty.$$

Proof. For two distinct candidates α and β, we have

$$\{|\alpha| \neq |\beta|\} = \left\{\frac{S - n\bar{w}1}{\sqrt{n}} \in G\right\},$$

where $G = \left\{s \in \mathbf{R}^C : s_\alpha \neq s_\beta\right\}$. By Proposition 2 (and Durrett (1996), p.87), since G is an open set we have

$$\liminf_n P\left(\frac{S - n\bar{w}1}{\sqrt{n}} \in G\right) = P\left(\sigma_w \left(\frac{m}{m-1}\right)^{1/2} (Z - \bar{Z}\mathbf{1}) \in G\right) = P\,(Z_\alpha \neq Z_\beta) = 1.$$

That is, $P\,(|\alpha| = |\beta|) \to 0$ as $n \to \infty$. The result follows since

$$P\left(\bigcup_{\alpha \neq \beta} \{|\alpha| = |\beta|\}\right) \leq \sum_{\alpha \neq \beta} P\,(|\alpha| = |\beta|).$$

∎

3 Approximations of the minimum manipulating coalition size

In this section, we formulate the coalitional manipulation problem as an integer linear program. We then show, via a series of simplifying steps, that this is well approximated by a much simpler linear program in which there are only two variables, and in which the constraint set does not depend on the voting situation, but only on the voting rule. We summarize the steps involved before embarking on them:

- The problem of assembling the smallest possible manipulating coalition can be expressed as an integer linear program, in which the variables are the numbers of voters of each type (x_t) to recruit;
- The integrality and upper bound ($x_t \leq N_t$) constraints of this program may be ignored (Proposition 4).
- Only manipulations in favour of the second-placegetter need be considered (Proposition 6).
- Coalition recruiting may be limited to those voters who rank the two leading candidates a and b adjacent (b above a), as these voters are best able to manipulate (Proposition 7).
- We may find the minimum coalition size by considering only the members' (sincere) rankings of a and b, without regard to how they rank other candidates (Theorem 8). This reduces the problem to a mere linear program with $m-1$ variables and two constraints.
- Replacing this linear program with its dual gives us two variables and $m-1$ constraints.

In the present paper, we will consider manipulation only at profiles (or voting situations) for which there is a sole winner a (i.e. a candidate a with $|a| > |\beta|$ for each $\beta \neq a$). Henceforth, we shall always assume that the profile at hand is of this kind. This is justified by Proposition 3 which tells us that the probability of a tied situation becomes 0 in the asymptotic limit. The reader interested in manipulation of tied situations is referred to Pritchard and Wilson (2007).

To specify an attempted coalitional manipulation, we must specify for each $t \in T$ the number x_t of coalition members who are recruited from (sincere preference) type t, as well as the number y_t of coalition members who will insincerely vote t. Of course, we must have $\sum_{t\in T} x_t = \sum_{t\in T} y_t$. Following a manipulation attempt, the revised score of a candidate α will be

$$|\alpha| - \sum_{t\in T} x_t\sigma_t(\alpha) + \sum_{t\in T} y_t\sigma_t(\alpha).$$

At a profile with sole winner a, a manipulation attempt is successful in favour of a candidate β if *(i)* the coalition members all prefer β to a, and *(ii)* the manipulated score of β matches or exceeds that of all other candidates, including a. (It is convenient to allow the possibility that β ties with other candidates for the manipulated win, although in the limit this makes no difference – requiring β to become the sole winner would not change any of our results for $n \to \infty$.)

It is clear that the coalition members need only consider insincere votes of types which rank β in first place; this is a dominant strategy for such manipulations. Let T_β be the set of such types. The manipulated score of any candidate α can then be written

$$|\alpha| - \sum_{t\in \bar{T}_{\beta a}} x_t\sigma_t(\alpha) + \sum_{t\in T_\beta} y_t\sigma_t(\alpha),$$

where $\bar{T}_{\beta a} \subseteq T$ is the set of types preferring β to a. In particular, the manipulated score of β will be

$$|\beta| - \sum_{t \in \bar{T}_{\beta a}} x_t \sigma_t(\beta) + \sum_{t \in T_\beta} y_t.$$

since $\sigma_t(\beta) = 1$ for $t \in T_\beta$.

The problem of constructing the smallest possible coalition (x_t) and associated strategy (y_t) for successful manipulation in favour of β can then be expressed by the following optimization problem.

$$\begin{array}{lcl} \min & \sum_{t \in \bar{T}_{\beta a}} x_t & \\ \text{s.t.} & \sum_{t \in T_\beta} y_t(1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{\beta a}} x_t(\sigma_t(\beta) - \sigma_t(\alpha)) \geq |\alpha| - |\beta| & \forall \alpha \neq \beta \\ & \sum_{t \in T_\beta} y_t = \sum_{t \in \bar{T}_{\beta a}} x_t & \\ & y_t \geq 0 & \forall t \in T_\beta \\ & 0 \leq x_t \leq N_t & \forall t \in \bar{T}_{\beta a} \\ & x_t, y_t \text{ integer} & \end{array} \tag{1}$$

Let $Q_1(\beta)$ be the value of (1) (or ∞ if (1) is infeasible). Let MCS be the minimum size of a successful manipulating coalition (or ∞ if no manipulation is possible). For completeness of definition, we let both of these random variables (and others we shall meet later) take the value ∞ when the voting situation N does not have a sole winner, or when $\beta = a$. Then

$$MCS = \min_{\beta \neq a} Q_1(\beta).$$

Note the last two constraints of (1) ($x_t \leq N_t$ and x_t, y_t integer): voters are discrete entities who must be recruited from among those actually available in the profile. We now aim to show that, in the asymptotic limit of a large voting population, these constraints are unimportant and may be dropped. This is as we should expect: each N_t will be about $n/m!$, while the differences between candidates' scores (and hence, presumably, coalition sizes) are likely to be of order only $\sqrt{n}$. Similarly, the requirement that x_t and y_t be integral should not be much of a hindrance when dealing with large numbers of voters.

To this end, let $Q_2(\beta)$ be the value of a linear program similar to (1), except that the constraints $x_t \leq N_t$, and the integrality constraints on x_t and y_t, are omitted.

Proposition 4.

$$P\left(|Q_1(\beta) - Q_2(\beta)| \leq K\right) \to 1 \qquad \text{as } n \to \infty,$$

where K is a constant that depends only on the voting rule. (In this and subsequent statements, a condition of the form $|q - p| \leq K$ is considered to hold if q and p are both infinite.)

Proof. See Appendix A.

Corollary 5.

$$P\left(|MCS - \min_{\beta \neq a} Q_2(\beta)| \leq K\right) \to 1 \qquad \text{as } n \to \infty$$

Proof. Follows immediately from the earlier observation that $MCS = \min_{\beta \neq a} Q_1(\beta)$. ∎

The above result simplifies the manipulation-strategist's task considerably. No longer is he limited by the numbers and preferences of the voters who happen to be present in the profile at hand. Instead, he may recruit an arbitrarily large (or even fractional) number of voters of any preference types (in $\bar{T}_{\beta a}$) he finds convenient. The resulting "coalition" may be smaller than the smallest coalition that can solve the manipulation problem as originally defined, but Corollary 5 assures us that it cannot be very much smaller.

Example. A 3-candidate election, with 8 voters, using Borda scoring ($w = (1, \frac{1}{2}, 0)$), produces the following voting situation:

2 voters have sincere preference acb (i.e. prefer a to c, and c to b);
2 voters have sincere preference abc;
3 voters have sincere preference bca;
1 voter has sincere preference cab;
no voters have preference bac or cba.

The Borda scores are $|a| = 4.5$, $|b| = 4$, and $|c| = 3.5$. Manipulation in favour of b is clearly not possible (i.e. $Q_1(b) = \infty$), because the only voters who prefer b to a already have b at the top and a at the bottom. But manipulation in favour of c is possible, with $Q_1(c) = 2$. The optimal solution to (1) in this case has $x_{bca} = y_{cba} = 2$, with all other x_t and y_t equal to zero. That is, two of the bca voters can insincerely vote cba, changing the result to a tie between a and c. (To make c the sole winner, all three bca voters would have to change their vote.)

In the more relaxed Q_2 sense, manipulation in favour of b becomes possible. We have $Q_2(b) = 1$, with an optimal solution having $x_{cba} = y_{bca} = 1$. (To make b the sole winner, we could have $x_{cba} = y_{bca} = 1.00001$.) The absence of the constraint $x_{cba} \le N_{cba}$ (where $N_{cba} = 0$) effectively allows the introduction of phantom voters who were never present in the original profile. Of course, this makes it more likely that "manipulation" will be possible. The essential point of Corollary 5 is that it does not become very much more likely, nor do the required coalition sizes change by very much.

We can simplify the problem still further. Let b be a candidate with second-highest score after a. The next result consists of the observation that only manipulations in favour of b need now be considered.

Proposition 6. $\min_{\beta \neq a} Q_2(\beta) = Q_2(b)$.

Proof. Let $(x_t)_{t \in \bar{T}_{\beta a}}$, $(y_t)_{t \in T_\beta}$ be optimal for the problem defining $Q_2(\beta)$, some $\beta \neq b$. Form $(x'_t)_{t \in \bar{T}_{\beta a}}$, $(y'_t)_{t \in T_\beta}$ by transposing β and b in all the voter types involved. (So if types s and t are related by transposition of the ranks of β and b, then $x'_s = x_t$.) We have, for any α,

$$\sum_{t \in T_b} y'_t(1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{ba}} x'_t(\sigma_t(b) - \sigma_t(\alpha)) = \sum_{t \in T_\beta} y_t(1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{\beta a}} x_t(\sigma_t(\beta) - \sigma_t(\alpha)).$$

Since $|\beta| \le |b|$, we see that (x'_t), (y'_t) are feasible for the problem defining $Q_2(b)$, and give the same objective value. The result follows. ∎

The reader should note that Proposition 6 does not hold for ordinary manipulation (i.e. with Q_1 substituted for Q_2), as the above example shows.

We next show that a coalition for the $Q_2(b)$ problem may always be formed by recruiting only voters who sincerely rank b and a adjacent (with b above a). Let $T_i \subset T$ consist of those types which rank b in ith place and a in $(i+1)$st place, and $T_{ba} = \cup_{i=1}^{m-1} T_i \subset \bar{T}_{ba}$. If we replace $\bar{T}_{ba}$ by T_{ba} in the linear program defining $Q_2(b)$, we obtain the linear program

$$\begin{array}{lcr} \min & \sum_{t \in T_{ba}} x_t & \\ \text{s.t.} & \sum_{t \in T_b} y_t(1 - \sigma_t(\alpha)) - \sum_{t \in T_{ba}} x_t(\sigma_t(b) - \sigma_t(\alpha)) \ge |\alpha| - |b| & \forall \alpha \neq b \\ & \sum_{t \in T_b} y_t = \sum_{t \in T_{ba}} x_t & \\ & x_t \ge 0 & \forall t \in T_{ba} \\ & y_t \ge 0 & \forall t \in T_b \end{array} \qquad (2)$$

Let Q denote the optimal value of (2) (or ∞ if (2) is infeasible).

Proposition 7. $Q_2(b) = Q$.

Proof. For $t \in T_i$, let $\tau(t) \subseteq \bar{T}_{ba}$ consist of those types which (i) agree with t in ranking positions $i+1, \ldots, m$; and (ii) rank the remaining candidates, other than b, in either the same position as t does, or one place lower. Observe that $\{\tau(t) : t \in T_{ba}\}$ is a partition of $\bar{T}_{ba}$, and that for $s \in \tau(t)$, $\alpha \neq b$

$$\sigma_t(b) - \sigma_t(\alpha) \leq \sigma_s(b) - \sigma_s(\alpha).$$

(This implies that a voter of type $s \in \tau(t)$ can always be dismissed from the manipulating coalition and replaced with one of type t.)

Let $(x_t)_{t \in \bar{T}_{ba}}$, $(y_t)_{t \in T_b}$ be optimal for the problem defining $Q_2(b)$. Form $(x'_t)_{t \in T_{ba}}$ by $x'_t = \sum_{s \in \tau(t)} x_s$. Then $\sum_{t \in T_{ba}} x'_t = \sum_{s \in \bar{T}_{ba}} x_s$, and

$$\sum_{t \in T_{ba}} x'_t(\sigma_t(b) - \sigma_t(\alpha) \leq \sum_{s \in \bar{T}_{ba}} x_s(\sigma_s(b) - \sigma_s(\alpha)).$$

Hence (x'_t) is feasible for (2). The result follows. ∎

Our efforts thus far have established that $P\left(|MCS - Q| \leq K\right) \to 1$ as $n \to \infty$. This will ensure that for IC asymptotic purposes, we may replace our original description of the minimum manipulating coalition size with the more tractable problem (2).

Further simplification is possible. The manipulation strategist solving the problem (2) must recruit voters of $m-1$ basic kinds, described by the type-sets $T_1, \ldots, T_{m-1}$. Our next result will show that what really matters are the numbers $z_1, \ldots, z_{m-1}$ of voters of each basic kind recruited. The allocation of the z_i voters of the ith kind to specific types within T_i is unimportant for our purposes. To that end, define the problem:

$$\begin{array}{rll} \min & \sum_{i=1}^{m-1} z_i & \\ \text{s.t.} & \sum_{i=1}^{m-1}(1 - w_i + w_{i+1}) z_i \geq |a| - |b| & \\ & \sum_{i=1}^{m-1}(1 - w_i) z_i \geq n\bar{w} - |b| & \\ & z_i \geq 0 & \text{for } i = 1, \ldots, m-1 \end{array} \tag{3}$$

where $\bar{w} = (w_1 + \cdots + w_m)/m$. Note that if z_i voters of types in T_i all cast insincere votes which rank b first, the score of b will be increased by $z_i(1 - w_i)$. If they all cast insincere votes which rank a last, the score of a will be decreased by $z_i w_{i+1}$. Thus, the first constraint of (3) makes it possible for b to catch up to a. The second constraint of (3) is less intuitive: it makes it possible for b to catch up to the average candidate's score $n\bar{w}$. This condition is clearly necessary if b is to win; it is also (roughly speaking) sufficient to ensure that while overtaking a, b is not himself overtaken by some third candidate. It is remarkable that these two (apparently rather weak) linear conditions should be all that is required to describe manipulation in the IC asymptotic limit.

Theorem 8. The linear programs (2) and (3) have the same optimal value.

Proof. See Appendix B.

Theorem 9. The linear programs (2) and (3) have the same objective value as the following one.

$$\begin{array}{rll} \max & (|a| - n\bar{w})\lambda + (n\bar{w} - |b|)\mu & \\ \text{s.t.} & w_{i+1}\lambda + (1 - w_i)\mu \leq 1 & \text{for } i = 1, \ldots, m-1 \\ & 0 \leq \lambda \leq \mu. & \end{array} \tag{4}$$

Also, (4) is unbounded if and only if (2) and (3) are infeasible.

Proof. The linear program (3) has the same objective value as its dual program (Bazaraa (2005), p. 251):

$$\begin{array}{rl} \max & (|a|-|b|)\lambda + (n\bar{w}-|b|)\lambda' \\ \text{s.t.} & (1-w_i+w_{i+1})\lambda + (1-w_i)\lambda' \leq 1 \quad \text{for } i=1,\ldots,m-1 \\ & \lambda \geq 0, \quad \lambda' \geq 0. \end{array}$$

Substituting $\mu = \lambda + \lambda'$ yields (4). ∎

Remark. We have now replaced our original description of the minimum manipulating coalition size with the simple two-variable linear program (4). This will be exploited in the remaining sections of the paper.

4 IC asymptotics of the minimum manipulating coalition size

The results of the previous section have established that

$$P\left(|MCS - Q| \leq K\right) \to 1 \qquad \text{as } n \to \infty,$$

where

$$Q = \max\left\{\lambda(|a| - n\bar{w}) + \mu(n\bar{w} - |b|) : (\lambda,\mu) \in M_w\right\}$$

and

$$M_w = \left\{(\lambda,\mu) : 0 \leq \lambda \leq \mu \text{ and } w_{i+1}\lambda + (1-w_i)\mu \leq 1 \text{ for } i = 1,\ldots,m-1\right\}.$$

Note in particular that the constraint set M_w does not depend on the voting situation, but only on the voting rule. For a given rule, it is possible to identify the corresponding M_w (a two-dimensional linear polytope), and then to identify the (finitely many) vertices of M_w which may achieve the optimum Q. This often leads to an explicit expression for Q in terms of $|a|$ and $|b|$.

Furthermore, we have $(|a|, |b|) = (\rho_1(S), \rho_2(S))$, where $\rho_j(x)$ denotes the jth largest element of a vector x. Proposition 2 gives us

$$\frac{(\rho_1(S) - n\bar{w}, n\bar{w} - \rho_2(S))}{\sqrt{n}} \xrightarrow{D} \sigma_w \left(\frac{m}{m-1}\right)^{1/2} (\rho_1(Z) - \bar{Z}, \bar{Z} - \rho_2(Z)),$$

and so it follows that

$$\frac{Q}{\sqrt{n}} \xrightarrow{D} V_w,$$

where

$$V_w = \max\left\{\lambda(\rho_1(Z) - \bar{Z}) + \mu(\bar{Z} - \rho_2(Z)) : (\lambda,\mu) \in \sigma_w \left(\frac{m}{m-1}\right)^{1/2} M_w\right\}.$$

Hence, too,

$$\frac{MCS}{\sqrt{n}} \xrightarrow{D} V_w,$$

by the converging-together lemma (Durrett (1996), p.91). Consequently,

$$P\left(MCS \leq v\sqrt{n}\right) \to g_w(v) := P\left(V_w \leq v\right) \qquad \text{as } n \to \infty.$$

That is, the asymptotic probability that the voting situation is manipulable by a coalition of $v\sqrt{n}$ or fewer voters is computable as a (non-decreasing) function of v. This function depends only on the voting rule.

A further observation will be helpful in determining which vertices of M_w may achieve the above maximum. If $x \in \mathbf{R}^m$ has mean element $\bar{x} = (x_1 + \cdots + x_m)/m$, then

$$\rho_1(x) - \bar{x} \geq 0 \qquad \text{and} \qquad -(\rho_1(x) - \bar{x}) \;\leq\; \bar{x} - \rho_2(x) \;\leq\; \frac{1}{m-1}(\rho_1(x) - \bar{x}).$$

We note in passing that for all rules other than the anti-plurality rule $w = (1, \ldots, 1, 0)$, M_w is a bounded set, since the constraints defining it include $(1 - w_{m-1})\mu \leq 1$. Hence V_w is a finite-valued random variable. It follows that $P\,(MCS = \infty) \to 0$ as $n \to \infty$, a well-known result (Kim and Roush (1996)).

The remainder of this section will be devoted to carrying out the above analysis for some common positional voting rules.

Borda's rule. $w_i = (m - i)/(m - 1)$ for $i = 1, \ldots, m$. The constraints defining M_w for this rule are

$$\begin{aligned}
(m-2)\lambda &\leq m-1 \\
(m-3)\lambda + \mu &\leq m-1 \\
&\vdots \\
\lambda + (m-3)\mu &\leq m-1 \\
(m-2)\mu &\leq m-1 \\
0 \leq \lambda &\leq \mu
\end{aligned}$$

Note that the point with $\lambda = \mu = (m-1)/(m-2)$ satisfies each of the constraints with equality. So M_w is the triangle with vertices at $(0,0)$, $(0, \frac{m-1}{m-2})$, and $(\frac{m-1}{m-2}, \frac{m-1}{m-2})$; the last of these always achieves the optimum Q. Thus, a good approximation to the minimum manipulating coalition size is

$$Q = \left(\frac{m-1}{m-2}\right)(|a| - |b|).$$

For the corresponding asymptotic result, note that

$$\sigma_w^2 = \frac{m+1}{12(m-1)}.$$

It then transpires that

$$\frac{MCS}{\sqrt{n}} \xrightarrow{D} \left(\frac{m(m+1)}{12(m-2)^2}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)).$$

Anti-plurality rule: $w = (1, \ldots, 1, 0)$. The constraints defining M_w reduce to $0 \leq \lambda \leq 1$, $\mu \geq \lambda$. Thus

$$Q = \begin{cases} |a| - |b|\,, & \text{if } |b| \geq n\bar{w} \\ \infty, & \text{otherwise.} \end{cases}$$

The corresponding asymptotic result is

$$\frac{MCS}{\sqrt{n}} \xrightarrow{D} V_w \;=\; \begin{cases} m^{-1/2}(\rho_1(Z) - \rho_2(Z)), & \text{if } \rho_2(Z) \geq \bar{Z} \\ \infty, & \text{otherwise.} \end{cases}$$

Uniquely among positional voting rules, anti-plurality may not admit the logical possibility of manipulation even in the IC asymptotic limit. This is reflected here by the possibility that

$V_w = \infty$. As a corollary of our asymptotic result, we can find the limiting probability that an anti-plurality election is invulnerable to manipulation:

$$\lim_n P\left(MCS = \infty\right) = P\left(V_w = \infty\right) = P\left(\rho_2(Z) < \bar{Z}\right),$$

a result essentially contained in Kim and Roush (1996).

Plurality and k-approval rules: $w = (1, \ldots, 1, 0, \ldots 0)$ (with k 1s), where $1 \le k \le m-2$. The simple plurality rule is included here as the case $k = 1$. The constraints defining M_w reduce to $0 \le \lambda \le \mu \le 1$, so M_w is the triangle with vertices at $(0,0)$, $(0,1)$, and $(1,1)$; the last of these always achieves the optimum. Thus, our approximation of the minimum manipulating coalition size is simply

$$Q = |a| - |b|.$$

While this expression is valid for all k-approval rules ($1 \le k \le m-2$), different values of k will give rise to different probability distributions for $(|a|, |b|)$, and so different asymptotic results for MCS. We have $\bar{w} = k/m$ and $\sigma_w^2 = k(m-k)/m^2$, giving

$$\frac{MCS}{\sqrt{n}} \xrightarrow{D} \left(\frac{k(m-k)}{m(m-1)}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)).$$

Three-candidate "easy case" rules: $w = (1, 1-q, 0)$ where $1/2 \le q \le 1$. This family includes the 3-candidate versions of plurality voting ($q = 1$) and Borda's rule ($q = 1/2$). The constraints defining M_w are $0 \le \lambda \le \mu$, $(1-q)\lambda \le 1$, and $q\mu \le 1$. Thus M_w is the triangle with vertices at $(0,0)$, $(0, q^{-1})$, and (q^{-1}, q^{-1}); the last of these always achieves the optimum. Thus

$$Q = q^{-1}(|a| - |b|).$$

We have $\sigma_w^2 = 2(1-q+q^2)/9$; it follows from this that

$$\frac{MCS}{\sqrt{n}} \xrightarrow{D} \left(\frac{1-q+q^2}{3q^2}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)).$$

Three-candidate "hard case" rules: $w = (1, 1-q, 0)$ where $0 < q \le 1/2$. This family includes the remaining 3-candidate positional rules not already considered. For these rules, M_w is the quadrilateral with vertices at $(0,0)$, $(0, q^{-1})$, $((1-q)^{-1}, (1-q)^{-1})$, and $(1-q)^{-1}, q^{-1})$. In voting situations with $|b| \ge n\bar{w}$, the optimum for Q is achieved at $((1-q)^{-1}, (1-q)^{-1})$; otherwise, $(1-q)^{-1}, q^{-1})$ is optimal. Thus

$$Q = \begin{cases} (1-q)^{-1}(|a| - |b|), & \text{if } |b| \ge n\bar{w} \\ (1-q)^{-1}(|a| - n\bar{w}) + q^{-1}(n\bar{w} - |b|), & \text{if } |b| \le n\bar{w} \end{cases}$$

that is

$$Q = (1-q)^{-1}(|a| - |b|) + \left(\frac{1}{q} - \frac{1}{1-q}\right)(n\bar{w} - |b|)_+,$$

where x_+ denotes $\max(x, 0)$. The corresponding asymptotic result is

$$\frac{MCS}{\sqrt{n}} \xrightarrow{D} \left(\frac{1-q+q^2}{3}\right)^{1/2} \left(\left(\frac{1}{1-q}\right)(\rho_1(Z) - \rho_2(Z)) + \left(\frac{1}{q} - \frac{1}{1-q}\right)(\bar{Z} - \rho_2(Z))_+\right).$$

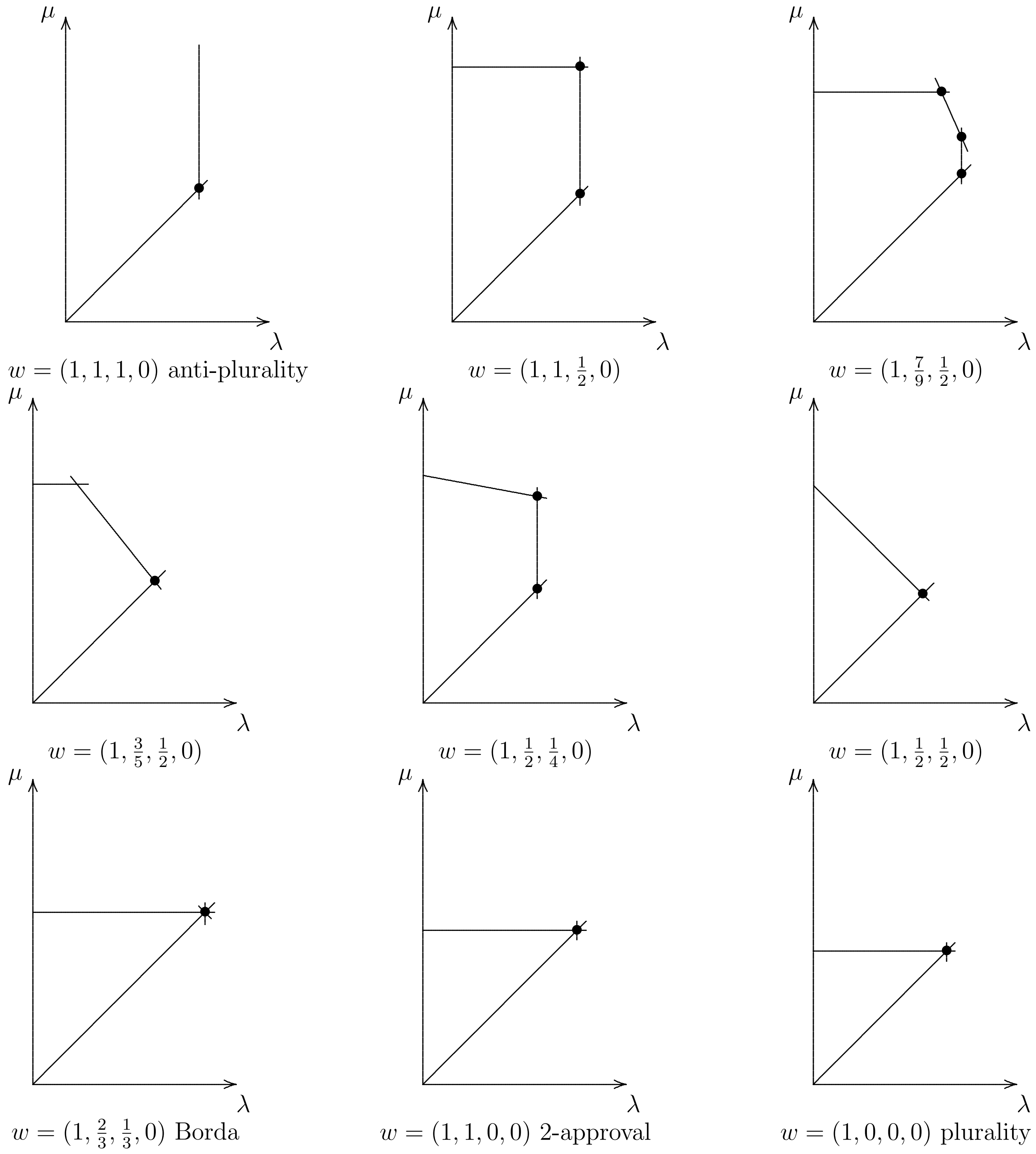

Figure 1: The sets $\sigma_w M_w$ for several four-candidate positional rules, depicted at consistent scales. The black dots show which points may be optimal.

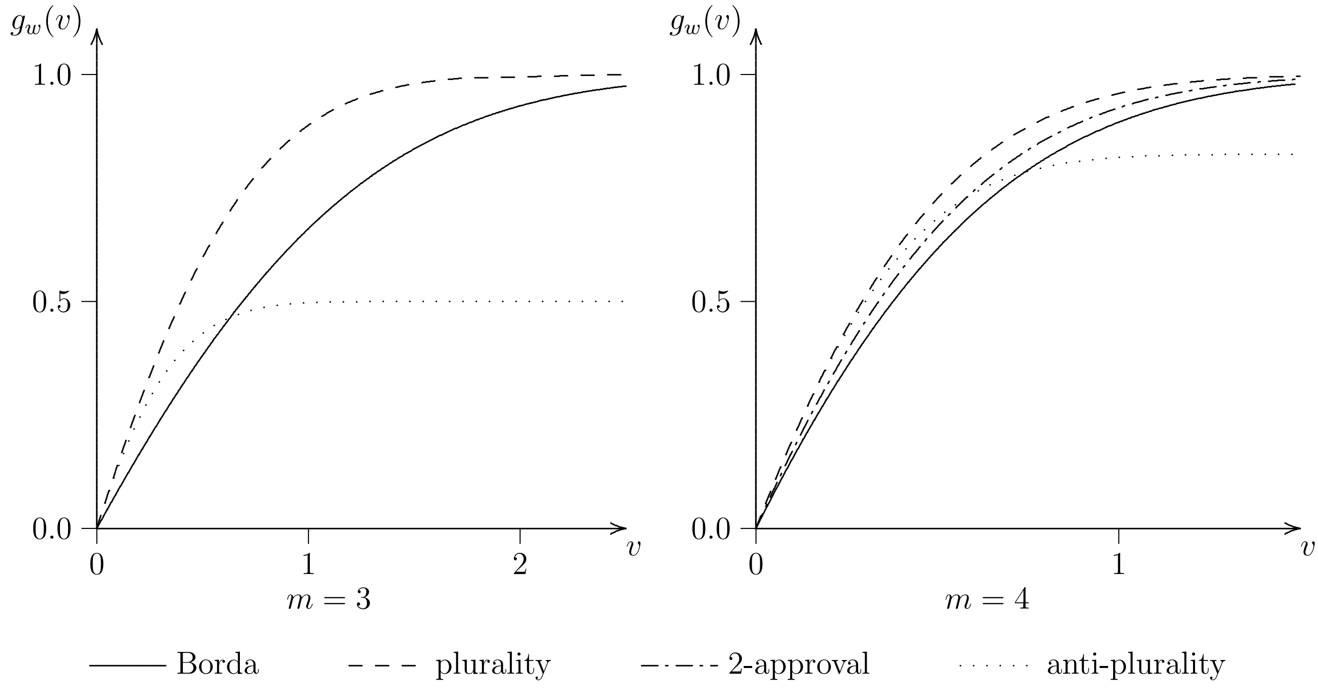

Figure 2: The functions $g_w(v) = P(V_w \le v)$ for some three- and four-candidate voting rules.

Four-candidate rules. For $m \ge 4$ it would be tedious to reduce all possible cases to asymptotic expressions of the kind above. Instead, we have simply illustrated the sets $\sigma_w M_w$ for a variety of rules w in Figure 1. Note that $\sigma_w M_w$ may have up to 3 optimal vertices. (For general m, $\sigma_w M_w$ might have up to $m-1$ optimal vertices.)

The functions $g_w(v) = P(V_w \le v)$ for some common voting rules are shown in Figures 2, 3, and 4.

Computing the quantities $g_w(v)$ requires evaluating integrals involving the normal probability density function $(2\pi)^{-1/2}e^{-x^2/2}$. This may be done by one of the standard methods of numerical quadrature, or, more simply, by Monte Carlo simulation using standard normal variates (Kleijnen and van Groenendaal (1992)). The latter approach has been used to produce Figures 2–4. The sample size used was 10^7 – large enough that the sampling error in the curves on the graphs is imperceptible to the eye.

5 Comparisons between positional voting rules

In this section we compare the various positional voting rules with respect to their manipulability under IC asymptotic conditions.

It is apparent from Figure 2 that the susceptibility of voting rules to coalitional manipulation depends on the size of the coalition involved. The graph for $m = 3$, for example, shows that elections using the plurality or Borda rules are highly likely to be manipulable by a large coalition (at least $2\sqrt{n}$ voters), whereas only about half of anti-plurality elections are so manipulable. But, if one is more concerned about manipulation by small groups, the graph shows that plurality and anti-plurality elections are about equally susceptible to manipulation by coalitions of less than $0.25\sqrt{n}$ voters, while Borda is rather less susceptible.

This suggests that there will be no single rule which is clearly superior to all others with respect to IC coalitional manipulation. However, some are clearly inferior to others. The asymptotic manipulation probability $g_w(v)$ for the plurality rule, for example, is greater than for the other two 3-candidate rules shown for all values of v.

More formally, given two m-candidate positional voting rules w, w', we will say that w *dominates* w' (and write $w' \preceq w$) if $g_w(v) \leq g_{w'}(v)$ for all $v \geq 0$. That is, w is less susceptible than w' to manipulation by coalitions of any given size. Alternatively, the asymptotic minimum coalition size V_w is larger than $V_{w'}$ in the sense of first-order stochastic dominance.

Note that $\preceq$ gives a partial order on the rules. Although the present paper is concerned only with positional rules, the partial order $\preceq$ could be defined in the same way for any voting rules.

If $w' \preceq w$, then the rule w is to be preferred to w'. More generally, the best rules to use (at least from the point of view of manipulation by IC populations) are those not dominated by any other.

Proposition 10. The plurality rule is always dominated.

Proof. Indeed, plurality is dominated both by anti-plurality and by Borda. This is apparent from the asymptotic results of the previous section:

$$V_{\text{antiplurality}} = \begin{cases} m^{-1/2}(\rho_1(Z) - \rho_2(Z)), & \text{if } \rho_2(Z) \geq \bar{Z} \\ \infty, & \text{otherwise ,} \end{cases}$$

while

$$V_{\text{plurality}} = m^{-1/2}(\rho_1(Z) - \rho_2(Z)),$$

giving $V_{\text{plurality}} \leq V_{\text{antiplurality}}$ and so (plurality) $\preceq$ (anti-plurality). Similarly for Borda. ∎

Proposition 11. The anti-plurality rule is never dominated.

Proof. No other rule w may dominate the anti-plurality rule, because $\lim_{v\to\infty} g_{\text{antiplurality}}(v) = P\left(\rho_2(Z) \geq \bar{Z}\right) < 1$, whereas $\lim_{v\to\infty} g_w(v) = 1$. ∎

Proposition 11 is true because the anti-plurality rule is resistant to manipulation by very large coalitions, in a way that every other positional rule is not. However, as we shall see later, this advantage becomes very slight in elections with 6 or more candidates.

Proposition 12. Borda's rule is undominated for $m \in \{3, 4\}$, but dominated for $m \geq 5$.

Proof. From the asymptotic results of the previous section:

$$V_{\text{Borda}} = \left(\frac{m(m+1)}{12(m-2)^2}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)),$$

while

$$V_{\lfloor m/2 \rfloor-\text{approval}} = \left(\frac{\lfloor m/2 \rfloor (m - \lfloor m/2 \rfloor)}{m(m-1)}\right)^{1/2} (\rho_1(Z) - \rho_2(Z)).$$

(Here $\lfloor \cdot \rfloor$ denotes the greatest integer function: thus $\lfloor m/2 \rfloor$ is $m/2$ if m is even, and $(m-1)/2$ if m is odd.) Note that for $m \geq 5$,

$$\frac{\lfloor m/2 \rfloor (m - \lfloor m/2 \rfloor)}{m(m-1)} \geq \frac{(m-1)(m+1)}{4m(m-1)} = 3\left(1 - \frac{2}{m}\right)^2 \frac{m(m+1)}{12(m-2)^2} \geq \frac{m(m+1)}{12(m-2)^2}.$$

This gives $V_{\text{Borda}} \leq V_{\lfloor m/2 \rfloor-\text{approval}}$ and so (Borda) $\preceq$ ($\lfloor m/2 \rfloor$-approval).

Now let $m \in \{3,4\}$ and let w be any positional rule for m-candidate elections. We may observe that

$$M_w \subseteq M'_w := \left\{(\lambda,\mu) \in \mathbf{R}^2 : 0 \leq \lambda \leq b_w, \lambda \leq \mu\right\},$$

where

$$b_w = \min_i (1 - w_i + w_{i+1})^{-1}.$$

It is shown in Pritchard and Slinko (2006) that the quantity $\sigma_w b_w$ is maximized, over all m-candidate positional rules, by Borda's rule. (This is true only when $m \in \{3,4\}$; for $m \geq 5$, the maximum value is achieved by the $\lfloor m/2 \rfloor$-approval rule.) From this we obtain

$$\begin{aligned}
g_w(v) &= P\left(V_w \leq v\right) \\
&\geq P\left(V_w \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&\geq P\left(\max\left\{\lambda(\rho_1(Z) - \bar{Z}) + \mu(\bar{Z} - \rho_2(Z)) : (\lambda,\mu) \in \sqrt{\tfrac{m}{m-1}}\sigma_w M_w\right\} \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&\geq P\left(\max\left\{\lambda(\rho_1(Z) - \bar{Z}) + \mu(\bar{Z} - \rho_2(Z)) : (\lambda,\mu) \in \sqrt{\tfrac{m}{m-1}}\sigma_w M'_w\right\} \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&= P\left(\sqrt{\tfrac{m}{m-1}}\sigma_w b_w(\rho_1(Z) - \rho_2(Z)) \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&\geq P\left(\sqrt{\tfrac{m}{m-1}}\sigma_{\text{Borda}} b_{\text{Borda}}(\rho_1(Z) - \rho_2(Z)) \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&= P\left(V_{\text{Borda}} \leq v \text{ and } \rho_2(Z) \geq \bar{Z}\right) \\
&= g_{\text{Borda}}(v) \;-\; P\left(V_{\text{Borda}} \leq v \text{ and } \rho_2(Z) < \bar{Z}\right).
\end{aligned}$$

Let $c = (m/(m-1))^{1/2}\sigma_{\text{Borda}} b_{\text{Borda}}$. Since all orderings of $Z = (Z_1, \ldots, Z_m)$ are equally likely,

$$\begin{aligned}
P\left(V_{\text{Borda}} \leq v \text{ and } \rho_2(Z) < \bar{Z}\right) &= m! P\left(Z_1 \geq Z_2 \geq \cdots \geq Z_m,\ c(Z_1 - Z_2) \leq v,\ \text{ and } Z_2 < \bar{Z}\right) \\
&\leq m! P\left(0 \leq \bar{Z} - Z_2 \leq Z_1 - Z_2 \leq v/c\right) \\
&= m! \int_0^{v/c} \int_0^x f(x,y)\, dx dy \\
&\leq \frac{1}{2} m! (\sup f) \left(\frac{v}{c}\right)^2 \\
&= O(v^2) \qquad\qquad \text{as } v \to 0,
\end{aligned}$$

where f is the (non-degenerate) bivariate normal probability density of $(Z_1 - Z_2, \bar{Z} - Z_2)$. It follows that

$$g'_w(0) \geq g'_{\text{Borda}}(0),$$

and hence that w cannot dominate Borda. ∎

The argument used in Proposition 12 shows that among all positional rules for three- or four-candidate elections, Borda's rule has the g function with the smallest derivative at the origin. This means that it is the most resistant to manipulation by very small coalitions. However, this property does not hold when there are five or more candidates. In that case, it is the $\lfloor m/2 \rfloor$-approval rule which enjoys maximal resistance to manipulation by very small coalitions. These results are similar to those of Pritchard and Slinko (2006), although the criterion considered there is the "average threshhold coalition size" rather than the minimum manipulating coalition size of the present paper.

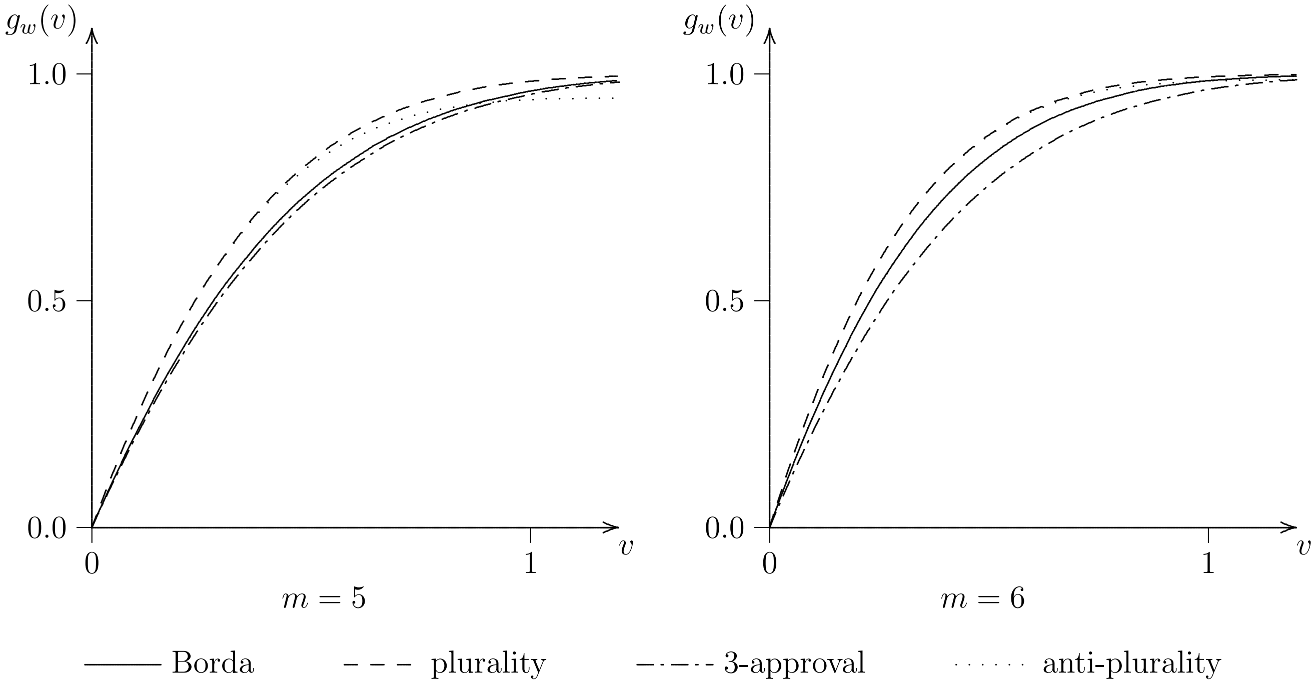

Figure 3: The functions $g_w(v) = P\left(V_w \leq v\right)$ for some five- and six-candidate voting rules.

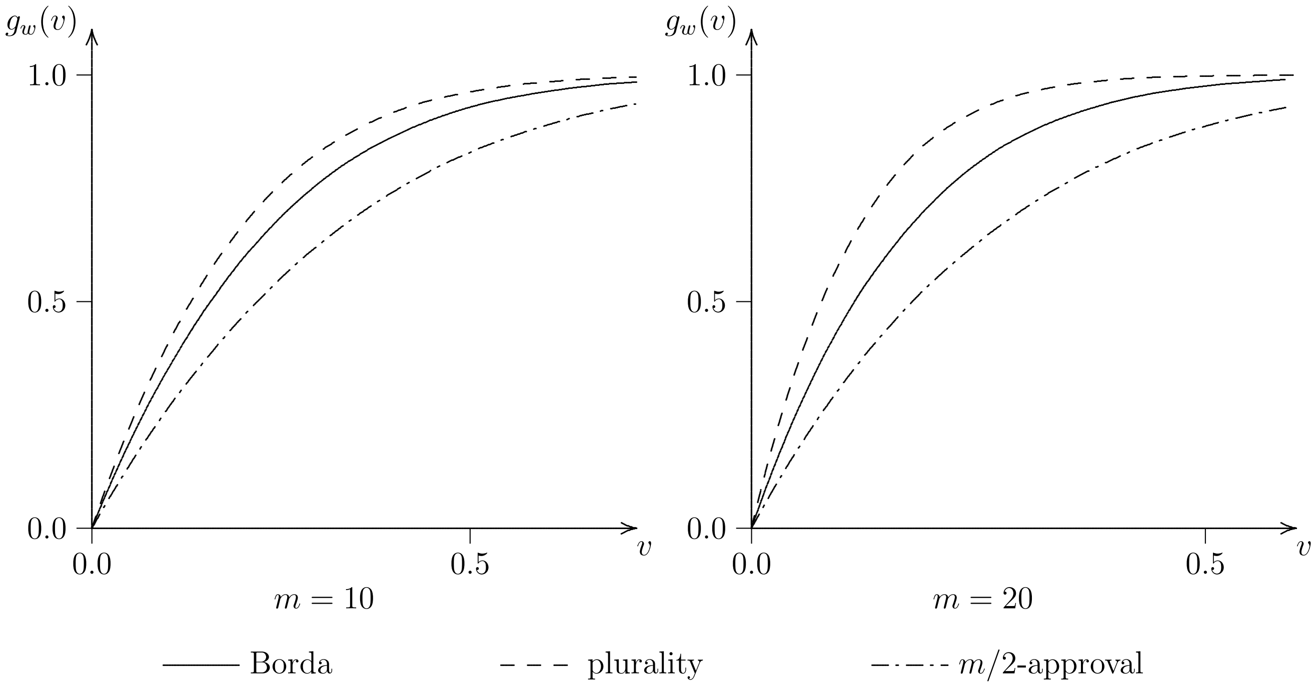

Figure 4: The functions $g_w(v) = P\left(V_w \leq v\right)$ for some ten- and twenty-candidate voting rules.

Figures 2–4 show the manipulability of the rules for particular numbers of candidates. We see from Figures 2 and 3 that for $m = 4, 5, 6$ there is not much difference in the manipulability of the common rules, at least by comparison with the differences evident when $m = 3$. In the graph for $m = 5$ we can see the Borda rule being dominated (by 3-approval) for the first time. (Note that for $m = 5$, the 2-approval and 3-approval rules have identical limiting behaviour.) Also worth noting is that for $m \geq 5$ there is very little difference between plurality and anti-plurality from a manipulation point of view. Anti-plurality has a slight additional chance of resisting attack by a large coalition – and on this basis dominates plurality – but this advantage has become almost imperceptible by $m = 6$.

Figure 4 shows the behaviour of the rules when there are many candidates. In this figure, the curves for the anti-plurality rule have been left out, as they are indistinguishable from those for plurality. We see the $\lfloor m/2 \rfloor$-approval rule dominating the others. It should be noted, though, that the IC hypothesis is at its least convincing when applied to elections with many candidates, as it assumes in effect that all candidates are about equally popular.

6 Conclusions

The technique presented in this paper makes it possible to compute the (IC) limiting probability distribution of the minimum manipulating coalition size for any positional rule, with any number of candidates.

The consideration of coalition sizes is especially useful when comparing the rules. Some rules are especially resistant to manipulation by small coalitions, while others fare better with respect to manipulation by large coalitions. Previous work has made this distinction in a rather limited way, by considering "individual" and "coalitional" manipulation (the latter meaning that the coalition may be of any size). But these extremes may be somewhat uninformative. Given a large voter population, all positional rules (except anti-plurality) are highly likely to be manipulable by some coalition, and highly unlikely to be manipulable by any individual. Studying coalitions of intermediate sizes starts to reveal more differences between the rules.

The picture also changes when the number of candidates is varied. Much previous work has concentrated on the three-candidate case, for which the behaviour of the rules is quite different (Borda is least susceptible to small-coalition manipulation, anti-plurality to large-coalition manipulation). The four-candidate case is similar, except that the differences between rules are smaller. But when there are five candidates, it appears that all positional rules are about equally manipulable, across the whole range of coalition sizes, and there is not much to choose between them. With six or more candidates, the $\lfloor m/2 \rfloor$-approval rules emerge as favourites.

Another, perhaps surprising, conclusion is that for $m \geq 5$ there is very little difference between plurality and anti-plurality from a manipulation point of view. The approximate symmetry between these rules does not appear when $m = 3$, and so does not appear to have been noticed before (although it was recognized in a more limited sense already in Saari (1990)).

It would be possible in principle to produce results like those in this paper for the IAC voter behaviour model. However, the technique of analysis would have to be quite different. Rather than reducing the probabilities to those involving normal distributions, the calculations would entail the computation of convex volumes, as outlined in Wilson and Pritchard (2007).

It would also be of interest to produce limiting distributions similar to those in this paper (or at least, graphs like those in Figures 2–4) for voting rules other than positional rules. However, this too would require new techniques. Approval voting (see Brams and Fishburn (2007)) would also be an attractive target, as it gives voters two potential ways to manipulate: in addition to mis-representing their preferences, they may also make a strategic choice as to the number of candidates to approve (a possibility investigated in Brams and Sanver (2006)).

Finally, it should be noted that this paper's assumptions on voter behaviour, though standard ones in the field, are very demanding with respect to manipulation. Situations harbouring the logical possibility of manipulation are made rather likely by the IC assumption, which tends to narrow margins of victory. Manipulation is further invited by the lack of credible counterthreats from voters outside the manipulating coalition. A valuable refinement of the present work would be to consider assumptions less favourable to manipulation. In particular, it would be useful to consider what strategies are (strong) Nash equilibria when all voters may vote strategically, and how much (if any) insincere voting they entail.

6.1 Acknowledgements

The authors wish to thank two anonymous reviewers for the example in section 3, and for other valuable comments.

7 Appendix A - Proof of Proposition 4

Let $Q_1(\beta)$ and $Q_2(\beta)$ be defined in the same way as in the main text, and let $Q_3(\beta)$ be defined in the same way as $Q_1(\beta)$, except that we drop the integrality constraints on x_t and y_t in (1), and replace the constraint $x_t \leq N_t$ by $x_t \leq N_t - K$, where K is a constant that depends only on the voting rule. We will choose the value

$$K = \begin{cases} 2m!(1-w_{m-1})^{-1}, & \text{if } w_{m-1} < 1 \\ 0, & \text{if } w_{m-1} = 1 \end{cases}$$

for K, although this choice will be important only in Proposition A.2. Note that we have $Q_2(\beta) \leq Q_1(\beta)$ and $Q_2(\beta) \leq Q_3(\beta)$.

Proposition A.1.

$$P\left(Q_3(\beta) = Q_2(\beta)\right) \to 1 \qquad \text{as } n \to \infty.$$

Proof. Define $f_0 : \mathbf{R}^T \to [0, \infty]$ as follows: for $q \in \mathbf{R}^T$, $f_0(q)$ is the optimal value of the linear program

$$\begin{array}{lcr} \min & \sum_{t \in \bar{T}_{\beta a}} x_t & \\ \text{s.t.} & \sum_{t \in T_\beta} y_t(1-\sigma_t(\alpha)) - \sum_{t \in \bar{T}_{\beta a}} x_t(\sigma_t(\beta) - \sigma_t(\alpha)) \geq \sum_{t \in T} q_t(\sigma_t(\alpha) - \sigma_t(\beta)) & \forall \alpha \neq \beta \\ & \sum_{t \in T_\beta} y_t = \sum_{t \in \bar{T}_{\beta a}} x_t, & \\ & x_t \geq 0 & \forall t \in \bar{T}_{\beta a} \\ & y_t \geq 0 & \forall t \in T_\beta \end{array}$$

Then $Q_2(\beta) = f_0(N)$. Note that $f_0(\lambda q + \mu e) = \lambda f_0(q)$ for any $q \in \mathbf{R}^T$, $\lambda \geq 0$, and $\mu \in \mathbf{R}$. Also, f_0 is continuous on the closed subset $L = \{q : f_0(q) < \infty\}$ of $\mathbf{R}^T$. Let $\pi : \mathbf{R}^T \to L$ be the projection which maps each point $q \in \mathbf{R}^T$ to the nearest point of L to q; then π is continuous and $\pi(\lambda q + \mu e) = \lambda \pi(q) + \mu e$ for any $q \in \mathbf{R}^T$, $\lambda \geq 0$, and $\mu \in \mathbf{R}$.

Let $f : \mathbf{R}^T \to [0, \infty)$ be given by $f(q) = f_0(\pi(q))$. Then f is continuous; has $f(\lambda q + \mu e) = \lambda f(q)$ for any $q \in \mathbf{R}^T$, $\lambda \geq 0$, and $\mu \in \mathbf{R}$; and $Q_2(\beta) = f(N)$ whenever $Q_2(\beta) < \infty$.

When $Q_2(\beta)=\infty$, we have $Q_3(\beta)=\infty$ too. On the other hand, when $Q_2(\beta)\le \min_t N_t - K$, the corresponding optimal point of the linear program for $Q_2(\beta)$ is also feasible for the linear program for $Q_3(\beta)$, so $Q_3(\beta)=Q_2(\beta)$. Hence

$$\{Q_3(\beta)\neq Q_2(\beta)\}\subseteq\{\min_t N_t - K < Q_2(\beta)<\infty\}\subseteq\{\min_t N_t - K < f(N)\}\,,$$

and so it suffices to show that this last event has probability converging to 0.

Now define $h:\mathbf{R}^T\to(-\infty,\infty)$ by $h(q)=(\min_t q_t)-f(q)$. By Proposition 1 and the continuity of h, we have

$$h\left(\frac{N-ne}{\sqrt{n}}\right)\overset{D}{\to} h(X),\qquad\qquad \text{where } X\sim N(0,\Sigma).$$

This yields

$$\frac{(\min_t N_t)-f(N)-n/m!}{\sqrt{n}}\overset{D}{\to} h(X),$$

from which

$$\limsup_n P\left(\frac{(\min_t N_t)-f(N)-n/m!}{\sqrt{n}}\le-\lambda\right)\le P\left(h(X)\le-\lambda\right)$$

for any $\lambda>0$. That is,

$$\limsup_n P\left((\min_t N_t)-K-f(N)\le n/m!-\lambda\sqrt{n}-K\right)\le P\left(h(X)\le-\lambda\right).$$

We have $n/m!-\lambda\sqrt{n}-K>0$ for sufficiently large n, so

$$\limsup_n P\left((\min_t N_t)-K\le f(N)\right)\le P\left(h(X)\le-\lambda\right).$$

Since λ was arbitrary,

$$P\left((\min_t N_t)-K\le f(N)\right)\to 0\qquad\qquad \text{as } n\to\infty,$$

and the result follows. ∎

Proposition A.2. With probability 1,

$$Q_1(\beta)\le Q_3(\beta)+K.$$

Proof of Proposition A.2 for the case $w_{m-1}<1$. Let $(x_t)_{t\in\bar{T}_{\beta a}}$, $(y_t)_{t\in T_\beta}$ be optimal for the problem defining $Q_3(\beta)$. Then

$$\begin{array}{lr}
\sum_{t\in T_\beta} y_t(1-\sigma_t(\alpha))-\sum_{t\in\bar{T}_{\beta a}} x_t(\sigma_t(\beta)-\sigma_t(\alpha))\ge|\alpha|-|\beta| & \forall\alpha\neq\beta\\
\sum_{t\in T_\beta} y_t=\sum_{t\in\bar{T}_{\beta a}} x_t & \\
0\le x_t\le N_t-K & \forall t\in\bar{T}_{\beta a}\\
y_t\ge 0 & \forall t\in T_\beta
\end{array}$$

Choose types $t_0\in\bar{T}_{\beta a}$, $t_1\in T_\beta$ such that t_0 ranks a last, β next-to-last, and some γ first, while t_1 is obtained from t_0 by transposing the rankings of β and γ. Define $(x'_t)_{t\in\bar{T}_{\beta a}}$, $(y'_t)_{t\in T_\beta}$ by $x'_{t_0}=x_{t_0}+K$, $y'_{t_1}=y_{t_1}+K$, and $x'_t=x_t$ and $y'_t=y_t$ for all other t. Then (x'_t), (y'_t) satisfy

$$\begin{aligned}
\sum_{t\in T_\beta} y'_t(1-\sigma_t(\alpha))\ &-\ \sum_{t\in\bar{T}_{\beta a}} x'_t(\sigma_t(\beta)-\sigma_t(\alpha))\\
&=\ \sum_{t\in T_\beta} y_t(1-\sigma_t(\alpha))-\sum_{t\in\bar{T}_{\beta a}} x_t(\sigma_t(\beta)-\sigma_t(\alpha))\\
&\qquad\qquad +K(1-w_{m-1})+K(\sigma_{t_0}(\alpha)-\sigma_{t_1}(\alpha))\\
&\ge\ |\alpha|-|\beta|+2m!
\end{aligned}$$

for each $\alpha \neq \beta$. Also, $\sum_{t \in T_\beta} y'_t = \sum_{t \in \bar{T}_{\beta a}} x'_t$ and $0 \leq x'_t \leq N_t$ for each t.

Now obtain $(x''_t)_{t \in \bar{T}_{\beta a}}$, $(y''_t)_{t \in T_\beta}$ by rounding each x'_t, y'_t to an integral value. The choice between rounding up and rounding down (i.e. between $x''_t = \lfloor x'_t \rfloor$ and $x''_t = \lceil x'_t \rceil$) can be made arbitrarily, but should be done in such a way that $\sum_{t \in T_\beta} y''_t = \sum_{t \in \bar{T}_{\beta a}} x''_t$. Since $\left| x''_t - x'_t \right| \leq 1$ and $\left| y''_t - y'_t \right| \leq 1$, we obtain

$$\sum_{t \in T_\beta} y''_t (1 - \sigma_t(\alpha)) - \sum_{t \in \bar{T}_{\beta a}} x''_t (\sigma_t(\beta) - \sigma_t(\alpha)) \geq |\alpha| - |\beta|,$$

and so (x''_t), (y''_t) are feasible for (1). We have $\sum_{t \in \bar{T}_{\beta a}} x''_t = \sum_{t \in \bar{T}_{\beta a}} x_t + K$, from which it follows that $Q_1(\beta) \leq Q_3(\beta) + K$. ∎

Proof of Proposition A.2 for the case $w_{m-1} = 1$. A separate proof is required for this case (the anti-plurality rule $w = (1, \ldots, 1, 0)$). We can show that $Q_1(\beta) \leq Q_3(\beta)$ by showing that the optimal (x_t), (y_t) for the problem defining $Q_3(\beta)$ are always integral (and hence give a feasible solution to (1)). To establish this, we will use a well-known result in linear programming (see, e.g. Papadimitriou and Steiglitz (1982) or Parker and Rardin (1988)), which assures us that the optimal solution of a linear program will always be integral when the constraint coefficient matrix A is totally unimodular (i.e. every square submatrix has determinant 1, −1, or 0).

A useful sufficient condition for total unimodularity is given in Papadimitriou and Steiglitz (1982) (Theorem 13.3) as follows: a matrix whose entries are all 1, −1, or 0 is totally unimodular if each column has at most two non-zero entries, and if the rows can be partitioned into two sets I_1 and I_2 such that: (i) if a column has two entries of the same sign, their rows are in different sets; (ii) if a column has two entries of different signs, their rows are in the same set.

For this problem A has columns corresponding to the variables x_t ($t \in \bar{T}_{\beta a}$) and y_t ($t \in T_\beta$). There is one row corresponding to each candidate $\alpha \neq \beta$, in which the entry in the column corresponding to x_t is -1 if t ranks α last, and 0 otherwise; the entry in the column corresponding to y_t is 1 if t ranks α last, and 0 otherwise. Let these rows constitute the set I_1. There is also a further row corresponding to the constraint $\sum_{t \in T_\beta} y_t - \sum_{t \in \bar{T}_{\beta a}} x_t = 0$; in this row, the entries in the columns corresponding to the x_t are all -1, and those corresponding to the y_t are all 1. Let this row constitute the set I_2. It is clear that this matrix satisfies the sufficient condition above; the result follows. ∎

From Propositions A.1 and A.2 we have

$$P\left(Q_1(\beta) - K \leq Q_3(\beta) \leq Q_2(\beta) \leq Q_1(\beta)\right) \to 1 \qquad \text{as } n \to \infty,$$

from which the conclusion of Proposition 4,

$$P\left(|Q_1(\beta) - Q_2(\beta)| \leq K\right) \to 1 \qquad \text{as } n \to \infty,$$

follows.

8 Appendix B - Proof of Theorem 8

To establish the equality of the optimal values of two optimization problems, it is sufficient to show that for each feasible point of the first problem, there is a corresponding feasible point of the second with the same objective value, and vice versa. The following two propositions use this approach to prove Theorem 8.

Proposition B.1. Suppose non-negative numbers $z_1, \ldots, z_{m-1}$ are feasible for (3). Then there exists $(x_t)_{t \in T_{ba}}$, $(y_t)_{t \in T_b}$ feasible for (2), with $\sum_{t \in T_i} x_t = z_i$.

Proof. Let D be the set of candidates other than a or b. For any type $t \in T$, we denote by $t(i)$ the candidate ranked in ith place by t.

We will write our proposed (x_t), (y_t) in terms of parameters r and $(u_\alpha)_{\alpha \in D}$, which are to be determined later in such a way that $0 \le r \le 1$, $u_\alpha \ge 0\ \forall \alpha$, and $\sum_{\alpha \in D} u_\alpha = 1$.

For each $\alpha \in D$, let

$$v_\alpha = \begin{cases} \frac{1-u_\alpha}{m-3} & \text{, if } m \ge 4 \\ 1 & \text{, if } m = 3; \end{cases}$$

note $\sum_{\alpha \in D} v_\alpha = 1$.

For each t, let $x_t = \sum_{k=1}^4 x_t^{(k)}$ and $y_t = \sum_{k=1}^4 t_t^{(k)}$, where

$$\begin{aligned} x_t^{(1)} &= \begin{cases} \frac{r u_{t(m)} z_i}{(m-3)!} & \text{, if } t \in T_i,\ i = 1, \ldots, m-2 \\ 0 & \text{, if } t \in T_{m-1} \end{cases} \\ y_t^{(1)} &= \begin{cases} \sum_{i=1}^{m-2} \frac{r u_{t(i+1)} z_i}{(m-3)!} & \text{, if } t \in T_b \text{ with } t(m) = a \\ 0 & \text{, otherwise} \end{cases} \end{aligned}$$

(this corresponds to $r u_\alpha z_i$ voters of types in T_i changing sincere votes of form $\ldots ba \ldots \alpha$ to insincere ones $b \ldots \alpha \ldots a$, for each $i = 1, \ldots, m-2$);

$$\begin{aligned} x_t^{(2)} &= \begin{cases} \frac{(1-r) v_{t(1)} z_i}{(m-3)!} & \text{, if } t \in T_i,\ i = 2, \ldots, m-2 \\ 0 & \text{, if } t \in T_1 \cup T_{m-1} \end{cases} \\ y_t^{(2)} &= \begin{cases} \frac{(1-r) v_{t(i)} z_i}{(m-3)!} & \text{, if } t \in T_b \text{ with } t(i+1) = a,\ i = 2, \ldots, m-2 \\ 0 & \text{, otherwise} \end{cases} \end{aligned}$$

(this corresponds to $(1-r) v_\alpha z_i$ voters of types in T_i changing sincere votes of form $\alpha \ldots ba \ldots$ to insincere ones $b \ldots \alpha a \ldots$, for each $i = 2, \ldots, m-2$);

$$x_t^{(3)} = y_t^{(3)} = \begin{cases} \frac{(1-r) z_1}{(m-2)!} & \text{, if } t \in T_1 \\ 0 & \text{, otherwise;} \end{cases}$$

(this corresponds to $(1-r) z_1$ voters of types in T_1 leaving their votes unchanged);

$$\begin{aligned} x_t^{(4)} &= \begin{cases} \frac{v_{t(1)} z_{m-1}}{(m-3)!} & \text{, if } t \in T_{m-1} \\ 0 & \text{, if } t \notin T_{m-1} \end{cases} \\ y_t^{(4)} &= \begin{cases} \frac{v_{t(m-1)} z_{m-1}}{(m-3)!} & \text{, if } t \in T_b \text{ with } t(m) = a \\ 0 & \text{, otherwise} \end{cases} \end{aligned}$$

(this corresponds to $v_\alpha z_{m-1}$ voters of types in T_{m-1} changing sincere votes of form $\alpha \ldots ba$ to insincere ones $b \ldots \alpha a$).

The reader may verify that

$$\sum_{t \in T_i} x_t = \sum_{k=1}^4 \sum_{t \in T_i} x_t^{(k)} = z_i \qquad \text{and} \qquad \sum_{t \in T_b} y_t = \sum_{k=1}^4 \sum_{t \in T_b} y_t^{(k)} = \sum_{i=1}^{m-1} z_i.$$

We now verify that the inequality constraints of (2) hold. Note that

$$\sum_{t \in T_{ba}} x_t \sigma_t(b) = \sum_{i=1}^{m-1} z_i w_i, \qquad \sum_{t \in T_{ba}} x_t \sigma_t(a) = \sum_{i=1}^{m-1} z_i w_{i+1},$$

and

$$\begin{aligned}
\sum_{t\in T_b} y_t\sigma_t(a) &= \sum_{k=1}^{4}\sum_{t\in T_b} y_t^{(k)}\sigma_t(a) \\
&= 0 \;+\; (1-r)\sum_{i=2}^{m-2} z_i w_{i+1} \;+\; (1-r)z_1 w_2 \;+\; 0 \\
&= (1-r)\sum_{i=1}^{m-1} z_i w_{i+1}.
\end{aligned}$$

Hence

$$\sum_{t\in T_b} y_t(1-\sigma_t(a)) - \sum_{t\in T_{ba}} x_t(\sigma_t(b)-\sigma_t(a)) = B + rA,$$

where $A = \sum_{i=1}^{m-1} z_i w_{i+1}$ and $B = \sum_{i=1}^{m-1} z_i(1-w_i)$. The inequality constraint for $\alpha = a$ in (2) thus reduces to

$$B + rA \geq |a| - |b|\,.$$

Now consider the other inequality constraints. We use the notation $\bar{w}_{-i,j,k}$ to denote the average of all elements of w other than the ith, jth, and kth (i.e. $(-w_i - w_j - w_k + \sum_{\ell=1}^{m} w_\ell)/(m-3)$), or 0 if $m=3$. Similarly $\bar{w}_{-i,j} = (-w_i - w_j + \sum_{\ell=1}^{m} w_\ell)/(m-2)$. For $\alpha \in D$ we have

$$\begin{aligned}
\sum_{t\in T_{ba}} x_t\sigma_t(\alpha) &= \sum_{k=1}^{4}\sum_{t\in T_{ba}} x_t^{(k)}\sigma_t(\alpha) \\
&= r\sum_{i=1}^{m-2} z_i\left(u_\alpha\cdot 0 + \left(\sum_{\gamma\neq\alpha} u_\gamma\right)\bar{w}_{-i,i+1,m}\right) \\
&\quad + (1-r)\sum_{i=2}^{m-2} z_i\left(v_\alpha\cdot 1 + \left(\sum_{\gamma\neq\alpha} v_\gamma\right)\bar{w}_{-1,i,i+1}\right) \\
&\quad + (1-r)z_1\bar{w}_{-1,2} \\
&\quad + z_{m-1}\left(v_\alpha\cdot 1 + \left(\sum_{\gamma\neq\alpha} v_\gamma\right)\bar{w}_{-1,m-1,m}\right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{t\in T_b} y_t\sigma_t(\alpha) &= \sum_{k=1}^{4}\sum_{t\in T_b} y_t^{(k)}\sigma_t(\alpha) \\
&= r\sum_{i=1}^{m-2} z_i\left(u_\alpha w_{i+1} + \left(\sum_{\gamma\neq\alpha} u_\gamma\right)\bar{w}_{-1,i+1,m}\right) \\
&\quad + (1-r)\sum_{i=2}^{m-2} z_i\left(v_\alpha w_i + \left(\sum_{\gamma\neq\alpha} v_\gamma\right)\bar{w}_{-1,i,i+1}\right) \\
&\quad + (1-r)z_1\bar{w}_{-1,2} \\
&\quad + z_{m-1}\left(v_\alpha w_{m-1} + \left(\sum_{\gamma\neq\alpha} v_\gamma\right)\bar{w}_{-1,m-1,m}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{t\in T_b} y_t(1-\sigma_t(\alpha)) \quad &- \quad \sum_{t\in T_{ba}} x_t(\sigma_t(b)-\sigma_t(\alpha)) = \\
& \sum_{i=1}^{m-1} z_i - ru_\alpha \sum_{i=1}^{m-2} z_i w_{i+1} + r(1-u_\alpha)\sum_{i=1}^{m-2} z_i(\bar{w}_{-i,i+1,m} - \bar{w}_{-1,i+1,m}) \\
& \qquad +(1-r)v_\alpha \sum_{i=2}^{m-2} z_i(1-w_i) + z_{m-1}v_\alpha(1-w_{m-1}) - \sum_{i=1}^{m-1} z_i w_i \\
= \quad & \sum_{i=1}^{m-1} z_i(1-w_i) - ru_\alpha \sum_{i=1}^{m-1} z_i w_{i+1} + rv_\alpha \sum_{i=1}^{m-2} z_i(1-w_i) \\
& \qquad +(1-r)v_\alpha \sum_{i=1}^{m-2} z_i(1-w_i) + z_{m-1}v_\alpha(1-w_{m-1}) \\
= \quad & (1+v_\alpha)B - ru_\alpha A.
\end{aligned}$$

The $(x_t), (y_t)$ that we have constructed will thus satisfy the constraints required to be feasible for (2), provided r and (u_α) can be chosen in such a way that

$$\begin{aligned}
B + rA &\geq |a| - |b| \\
(1+v_\alpha)B - ru_\alpha A &\geq |\alpha| - |b| \qquad\qquad \forall \alpha \in D \\
\sum_{\alpha\in D} u_\alpha &= 1 \\
u_\alpha &\geq 0 \qquad\qquad \forall \alpha \in D \\
0 \leq r &\leq 1.
\end{aligned}$$

Suppose for the moment that $m \geq 4$. Then the inequality required of u_α may be written

$$\left(1 + \frac{1-u_\alpha}{m-3}\right) B - ru_\alpha A \geq |\alpha| - |b|\,,$$

or

$$\left(rA + \frac{B}{m-3}\right) u_\alpha \leq |b| - |\alpha| + \left(\frac{m-2}{m-3}\right) B,$$

a simple upper bound on u_α. Note that the upper bound is non-negative. For given r, it will be possible to choose non-negative (u_α) such that $\sum_{\alpha\in D} u_\alpha = 1$ while complying with these upper bounds if and only if

$$\sum_{\alpha\in D}\left(|b| - |\alpha| + \left(\frac{m-2}{m-3}\right) B\right) \geq rA + \frac{B}{m-3},$$

or

$$rA \leq (m-2)\,|b| - \sum_{\alpha\in D} |\alpha| + \left(\frac{(m-2)^2-1}{m-3}\right) B,$$

that is

$$rA \leq (m-1)(|b| + B - n\bar{w}) + |a| - n\bar{w}, \tag{$*$}$$

using the fact that the sum of all the candidates' scores is $mn\bar{w}$.

Consider now the case $m=3$. Then the sole $\alpha \in D$ has $u_\alpha = v_\alpha = 1$, and the inequality required of u_α reduces to

$$2B - rA \geq |\alpha| - |b|\,,$$

which is $(*)$.

We have thus reduced our requirements to a condition on $r \in [0,1]$:

$$|a| - |b| - B \ \le \ rA \ \le \ (m-1)(|b| + B - n\bar{w}) + (|a| - n\bar{w}).$$

To see that there exists $r \in [0,1]$ satisfying this condition we note that, firstly,

$$(m-1)(|b| + B - n\bar{w}) + (|a| - n\bar{w}) \ge 0;$$

secondly,

$$|a| - |b| - B \le A$$

(from the first constraint of (3)); and thirdly,

$$|a| - |b| - B \le (m-1)(|b| + B - n\bar{w}) + (|a| - n\bar{w}).$$

This last condition can be simplified to

$$m(|b| + B - n\bar{w}) \ge 0,$$

the second constraint of (3). ∎

Proposition B.2. Suppose $(x_t)_{t\in T_{ba}}$, $(y_t)_{t\in T_b}$ are feasible for (2). Then $z_1, \dots, z_{m-1}$ given by $z_i = \sum_{t\in T_i} x_t$ are feasible for (3).

Proof. The inequality for $\alpha = a$ in (2) says that

$$\sum_{t\in T_b} y_t(1 - \sigma_t(a)) \ge |a| - |b| + \sum_{t\in T_{ba}} x_t(\sigma_t(b) - \sigma_t(a)).$$

Noting that $\sum_{t\in T_b} y_t \sigma_t(a) \ge 0$, we obtain

$$\sum_{t\in T_b} y_t \ge |a| - |b| + \sum_{i=1}^{m-1} \sum_{t\in T_i} x_t(w_i - w_{i+1})$$

and so

$$\sum_{i=1}^{m-1} z_i \ge |a| - |b| + \sum_{i=1}^{m-1} (w_i - w_{i+1}) z_i$$

from which the first constraint of (3) can be seen to hold.

If we add the inequalities in (2) for all $\alpha \ne b$, we obtain

$$\sum_{t\in T_b} y_t \sum_{\alpha\ne b} (1 - \sigma_t(\alpha)) \ge \left(\sum_{\alpha\ne b} |\alpha| \right) - (m-1)|b| + \sum_{t\in T_{ba}} x_t \sum_{\alpha\ne b} (\sigma_t(b) - \sigma_t(\alpha)),$$

or

$$\sum_{t\in T_b} y_t \sum_{i=1}^{m-1} (1 - w_i) \ge nm\bar{w} - m|b| + \sum_{i=1}^{m-1} \sum_{t\in T_i} x_t \sum_{j\ne i} (w_i - w_j),$$

which gives

$$(1 - \bar{w}) \sum_{i=1}^{m-1} z_i \ge n\bar{w} - |b| + \sum_{i=1}^{m-1} z_i(w_i - \bar{w}),$$

from which the second constraint of (3) can be seen to hold. ∎

References

F. Aleskerov and E. Kurbanov, *On the degree of manipulability of social choice rules,* Automation and remote control 59 (1998), 1460–1468.

E. Baharad and Z. Neeman, *The asymptotic strategy-proofness of majority and Borda rules,* Review of Economic Design 7 (2002), 331–340.

M. Bazaraa, J. Jarvis, and H. Sherali, *Linear programming and network flows,* Wiley, 2005.

S. Brams and P. Fishburn, *Approval voting,* Springer, 2007.

S. Brams and R. Sanver, *Critical strategies under approval voting: Who gets ruled in and ruled out,* Electoral Studies 25 (2006), 287–305.

S. Berg and D. Lepelley, *On probability models in voting theory,* Statistica Neerlandica 48 (1994), 133–146.

J. Chamberlin, *Investigation into the relative manipulability of four voting systems,* Behavior Science 30 (1985), 195–203.

R. Durrett, *Probability: Theory and examples,* Duxbury, 1996.

A. Gibbard, *Manipulation of voting schemes: a general result,* Econometrica 41 (1973), 587–601.

B. G. Ju, *A characterization of plurality-like rules based on non-manipulability, restricted efficiency, and anonymity,* International Journal of Game Theory 33 (2005), 335–354.

J. S. Kelly, *Almost all social choice rules are highly manipulable, but a few aren't,* Social Choice and Welfare 10 (1993), 161–175.

K. H. Kim and F. W. Roush, *Statistical manipulability of social choice functions,* Group Decision and Negotiation 5 (1996), 263–282.

J. Kleijnen and W. van Groenendaal, *Simulation: a statistical perspective,* Wiley, 1992.

D. Lepelley and B. Mbih, *The proportion of coalitionally unstable situations under the plurality rule,* Economic Letters 24 (1987), 311–315.

D. Lepelley and B. Mbih, *The vulnerability of four social choice functions to coalitional manipulation of preferences,* Social Choice and Welfare 11 (1994), 253–265.

S. Maus, J. Peters, and T. Storcken, *Minimal manipulability: Unanimity and nondictatorship,* Journal of Mathematical Economics 43 (2007), 675–691.

S. Nitzan, *The vulnerability of point-voting schemes to preference variation and strategic manipulation,* Public Choice 47 (1985), 349–370.

C. Papadimitriou and K. Steiglitz, *Combinatorial optimization,* Prentice-Hall, 1982.

G. Parker and R. Rardin, *Discrete Optimization,* Academic Press, 1988.

G. Pritchard and A. Slinko, *On the average minimum size of a manipulating coalition,* Social Choice and Welfare 27 (2006), 263–277.

G. Pritchard and M. Wilson, *Exact results on manipulability of positional voting rules,* Social Choice and Welfare 29 (2007), 487–513.

M. Regenwetter et al, *Behavioral Social Choice,* Cambridge University Press, 2006.

W. Riker, *The theory of political coalitions,* Yale University Press, 1962.

W. Riker, *The art of political manipulation,* Yale University Press, 1986.

D. Saari, *Susceptibility to manipulation,* Public Choice 64 (1990), 21–41.

M. A. Satterthwaite, *Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions,* Journal of Economic Theory 10 (1975), 187–217.

D. A. Smith, *Manipulability measures of common social choice functions,* Social Choice and Welfare 16 (1999), 639–661.

M. Wilson and G. Pritchard, *Probability calculations under the IAC hypothesis,* Mathematical Social Sciences 54 (2007), 244–256.