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The Centroid is a Reference Point for the Symmetric Difference in d Dimensions

Gerald Weber

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Software Engineering
University of Auckland
Private Bag 92019
Auckland 1
New Zealand
www.se.auckland.ac.nz
info@se.auckland.ac.nz

The Centroid is a Reference Point for the Symmetric Difference in d Dimensions

Gerald Weber

Department of Computer Science

The University of Auckland

gerald@cs.auckland.ac.nz

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Abstract

It is shown that the centroid is a reference point for the symmetric difference with loss factor $c_z = 1 + 2\frac{d^2}{d+1} < 1 + 2d$ for convex figures under affine transformations. This generalizes an earlier result for two dimensions.

1 Introduction

We generalize the result of [3] to $d > 1$ dimensions. We show that the centroid is a reference point for the symmetric difference with loss factor $c_z = 1 + 2\frac{d^2}{d+1} < 1 + 2d$ for convex figures under affine transformations.

Reference point methods are so called approximate matching methods, i.e. they yield matches which differ only by a constant factor from the optimal match. In the case of exact matches they are therefore guaranteed to yield the exact result. Reference point methods reduce the degrees of freedom of the underlying matching problem and therefore yield very efficient algorithms.

In the foundational works on reference points Alt, Behrends and Blömer [1] and Alt, Aurenhammer and Rote [2] identified the advantages of reference point methods and presented strong results, e.g. that the Steiner Point is a reference point for the Hausdorff metric. In [3] Alt, Fuchs, Rote and Weber obtained sharp bounds for the centroid and the symmetric difference of convex figures in two dimensions and discussed algorithms utilizing the reference point. The results have been discussed e.g. in [4, 5, 6].

2 Main Result

In the following we assume $d > 1$, since for $d = 1$ the centroid is trivially a reference point with optimal loss factor 1.

Given a set Φ of figures, i.e. compact subsets of R^d, a metric $\delta(F,G) : \Phi \times \Phi \to R$, a set of transformations closed under composition $T \subset (R^d)^{R^d}$.

Be $\delta_T^{opt}(F,G) = \min_{t \in T} \delta(F, t(G))$. It is well known, δ_T^{opt} fulfils the triangle inequality. We identify points with vectors and translations, e.g. $a - b$ is the translation that maps b to a. It is also the translation vector. We number the directions of the axes and coordinates in R^d from 1 to d, i.e. we say e.g. direction d and d-coordinate. A *characteristic point* r for Φ is a function $r : \Phi \to R^d$.

Definition 1 *A* reference point r *with* loss factor c *for* δ, Φ *and* T *is a characteristic point for* Φ *with*

- r *is equivariant for* T*: for all* $t \in T$: $rt = tr$,
- *For all* F, G *there is a* $t \in T$ *with* $r(F) = r(t(G))$ *so that* $\delta(F, t(G)) \leq c\delta_T^{opt}(F, G)$.
  t *is called a* c-approximate match*.*

For a reference point r let $\delta^r(F,G) = \min_{t \in T, r(F) = r(t(G))} \delta(F, t(G))$.
Be $\text{loss}_T(F,G) = \frac{\delta^r(F,G)}{\delta_T^{opt}(F,G)}$.
Let z be the centroid of a figure. z is a characteristic point. Let $\delta_s(F,G)$ be the area of the symmetric difference $F \Delta G$. Let A be the set of affine transformations.

Let C be the set of convex figures.

Theorem 1 z *is a reference point with loss factor* $c_z = 1 + 2\frac{d^2}{d+1}$ *for* δ_s, C*, and the translations.*

Remark: Note that with the upcoming Theorem 3 z is also a reference point for affine transformations.

2.1 Preliminaries of the Proof

The *height* of a figure shall be the difference of minimal and maximal d-coordinates of points in F.
The *bottom* shall be the $d-1$ dimensional plane of axes except the direction d.
The *shadow* of a figure shall be the projection along direction d onto the bottom. It is a figure in $d-1$ dimensions; we call its $d-1$-volume an *area*.
For a figure F let F_s be the following figure: In a kind of Cavalieri principle, think of F as being made of infinitesimally thin sticks which are parallel to the d-axis. Stamp F so that all sticks start at the bottom. The resulting figure is F_s; the upper hull of F_s is the upper hull of F minus the lower hull of F.
The *thickness* of a figure shall be the height of F_s.
A *non-oblique* figure is a figure, where the thickness is equal to the height.
A *shear-to-fit* for figure F is a shear operation q parallel to direction d so that $q(F)$ is non-oblique.
A *pyramid* is the convex hull of a d-1 dimensional set and a not coplanar point.

Lemma 1 *For all convex figures* F *there is a shear-to-fit.*

Proof: Let F_O be the open set, for which F is the compact closure. Let $]ab[$ be one of the longest thin stick of F in direction d. Let $F'_O = F_O - a + b$. F_O and F'_O are disjoint, but b is a common limit of both open sets. Hence there is a hyperplane h separating F_O and F'_O (See Figure 1). It must go through b and therefore be a tangent to both sets. h can not be parallel to the d-axis due to the position of F_O and F'_O. $h - b + a$ is also a tangent of F_O and parallel to h. The shear operation q that makes h parallel to the bottom fulfils the demands. □

Lemma 2 F_s *is convex for all convex figures* F.

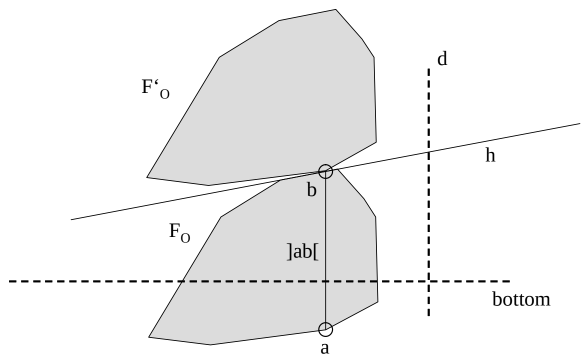

Figure 1:

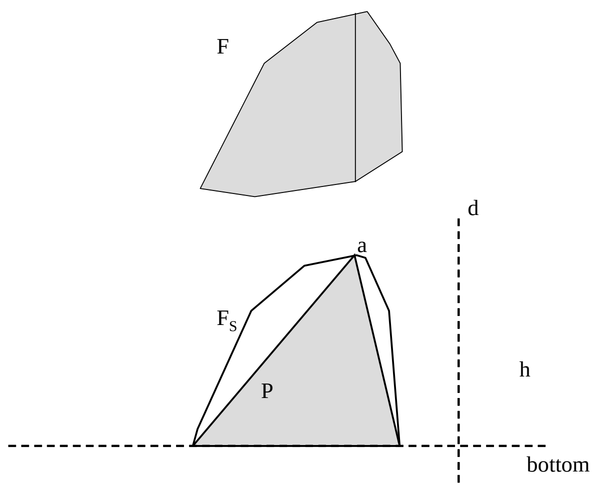

Figure 2:

Proof: As said, the upper hull of F_s is the difference of upper and lower hull of F. The lemma follows from the fact, that the sum of two convex functions is convex. □

Lemma 3 *For all figures with thickness 1 and an area of the shadow of 1, the pyramid has mimimum volume, namely* $1/d$*.*

Proof: The volume formula for the pyramid is well known. For the rest of the claim, let F be a figure with thickness 1 and an area of the shadow of 1, as shown in Figure 2 The thickness is also the height of F_s. The shadows and volumes of F and F_s are equal. Let a be a point of F_s with maximal d coordinate. Then the convex hull of the shadow of F and a is a pyramid that lies within F_s. □

Lemma 4 *Of all non-oblique convex figures of height* 1*, which touch the bottom from above, the standing pyramid has minimal* d*-coordinate of the centroid in dimension* d*. The centroid of the pyramid has* d*-coordinate* $1/(d+1)$*.*

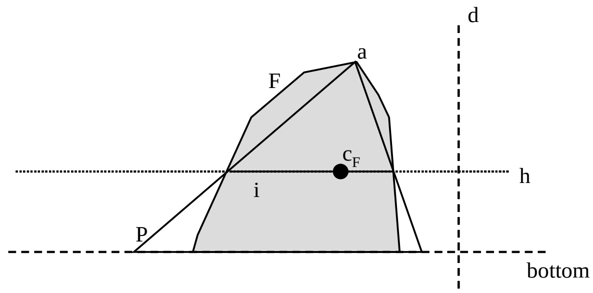

Figure 3:

In fact the shape of the pyramid's bottom facet does not matter.

Proof: The formula for the d-coordinate of the pyramid's centroid is well known.
We assume wlog. $F = F_s$ (using also Lemma 2). Then F stands on the bottom.
Let a be the top of F. We cut F at the height of its centroid with a $d-1$-dimensional hyperplane h parallel to the bottom. Let i be the intersection (See Figure 3) Let P' be the pyramid which is the convex hull of i and a. Let P be the smallest pyramid standing on the bottom, which contains P'. Due to convexity:

- F must contain the pyramid P'.
- The part of F below h must be contained within P, because the boundary of i is also boundary of F and P.

Hence $P - F$ lies below h and $F - P$ lies above h. From this follows immediately, that $z(P)$ has a smaller d-coordinate than $z(F)$. $\square$

Lemma 5 *Let be $\epsilon > 0$. There are F', G' with $F' \subset G'$ and $\delta^{opt}(F', G') = |G' - F'| < \epsilon$ so that* $\text{loss}_T(F', G') \geq \text{loss}_T(F, G)$.

Proof: It is a straightforward consequence of the triangle inequality. let wlog. be F, G in optimal position. Choose $F = F_1 \supset F_2 \supset \ldots \supset F_n = (F \cap G) \subset F_{n+1} \subset \ldots \subset F_m = G$, so that all $\delta_s(F_i, F_{i+1}) < \epsilon$, like in Figure 4. Because $F_i \Delta F_{i+1}$ are disjoint we have $\sum \delta_s(F_i, F_{i+1}) = \delta_s(F, G)$. The triangle inequality gives:

$$\sum \delta^r(F_i, F_{i+1}) \geq \delta^r(F, G)$$

Hence with $\text{loss}_T(F, G) = l$

$$\frac{\sum \delta^r(F_i, F_{i+1})}{\sum \delta_s(F_i, F_{i+1})} \geq \frac{\delta^r(F, G)}{\delta_s(F, G)} = l$$

We use the following fact about the the weighted mean as a generalized pigeon hole principle:

For $a_1, \ldots a_n$ and positive $b_1 \ldots b_n$ there is a $1 \leq j \leq n$ with $\frac{a_j}{b_j} \geq \frac{\sum a_i}{\sum b_i}$

From this fact follows, that there are F_j, F_{j+1}, with

$$\frac{\delta^r(F_j, F_{j+1})}{\delta_s(F_j, F_{j+1})} \geq l$$

Hence $F' = F_j, G' = F_{j+1}$ fulfil the lemma. $\square$

Lemma 6 *For every $t \in A$ (the affine mappings) : loss$_A(F, G) =$ loss$_A(t(F), t(G))$.*

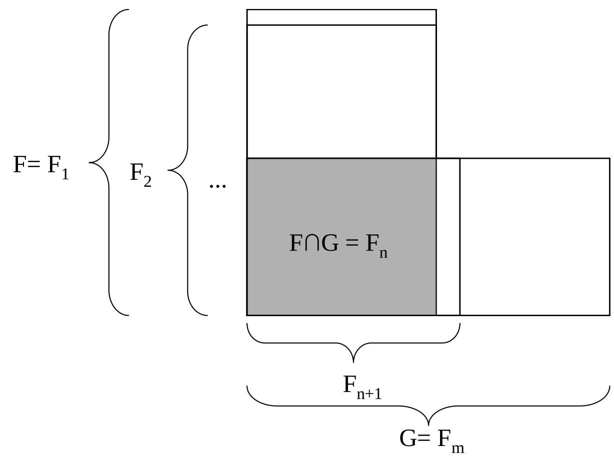

Figure 4:

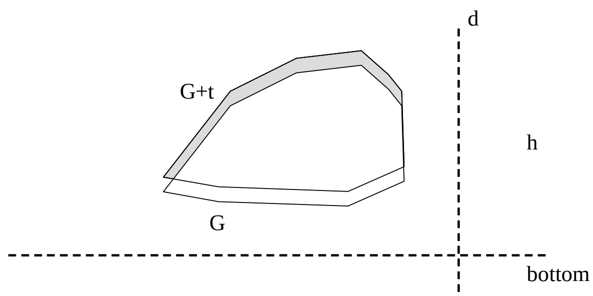

Figure 5: The displacement regions. One region is shaded.

Proof: The claim holds due to the equivariance of the centroid under affine mappings and the fact, that affine mappings preserve the ratio of areas. □

2.2 Proof of Theorem 1

The proof can be seen as basically showing, that there is nothing worse than the worst case which will be presented in theorem 2.

Because of Lemma 5, it suffices to consider $F \subset G$. Because of Lemma 6 we have enough degrees of freedom in order to assume wlog that F, G are normalized in the following way: $z(F) - z(G)$ is parallel to the d-axis, G is non-oblique (using Lemma 1) and has height 1 and an area of the shadow of 1. Using again Lemma 5 it suffices to consider $|G - F| < 1/d$.

Let $f = G - F, r = |f|/|G|, t = z(F) - z(G), k = |t|$. We will show, that the translation t is a c-approximate match. The set $G\Delta(t + G)$ consists of two regions as depicted in Figure 5, we call them the two displacement regions. As shown in [3] we know that each displacement region has area smaller $1 \cdot k$. Hence we have:

$$\delta_s(F, t + G) \leq 2k + |f| \quad (1)$$

With Lemma 3 we have

$$r \leq d|f| \quad (2)$$

Let $l = |z(f) - z(G)|$. With Lemma 4 we have, since $z(f)$ must be in G

$$l \leq \frac{d}{d+1} \tag{3}$$

The relation between the d coordinates of the centroids of F, f and G is described by the law of the lever.

$$|z(F) - z(G)| \cdot |F| = |z(f) - z(G)| \cdot |f|$$

$$k(1-r) = lr \tag{4}$$

Applying (3) gives:

$$k = l\frac{r}{1-r} \leq \frac{d}{d+1}|f|\frac{d}{1-d|f|} = |f|\frac{d^2}{(d+1)(1-d|f|)}. \tag{5}$$

The term $(1 - d|f|)$ in the denominator is now the last obstacle for proving the loss factor c_z from the theorem.

We will therefore first deduce an inequality similar to the desired one but with a disturbed version of the term we want to prove as loss factor $\tilde{c}(d,a) = 1 + 2\frac{d^2}{(d+1)(1-da)}$. Applying (1) and (5) we get:

$$\delta_s(F, t+G) \leq 2\frac{d^2}{(d+1)(1-d|f|)}|f| + |f| = (1 + 2\frac{d^2}{(d+1)(1-d|f|)})|f| = \tilde{c}(d,|f|)\delta_{opt}(F,G) \tag{6}$$

It is clear that $\tilde{c}(d,a) > c_z$. It is furthermore clear that $\lim_{a \to 0} \tilde{c}(d,a) = c_z$. The disturbed loss factor would be greater than the one we want to prove. Now we show with an asymptotic argument, that the disturbed loss factor cannot occur as an actual loss. We show by contradiction that there is no $F \subset G$ with loss$(F,G) > c_z$ in the following way: Assume F,G with loss$(F,G) = c' > c_z$.
Choose a so that $\tilde{c}(d,a) < c'$. Then by Lemma 5 there are F', G' with $F' \subset G'$, $\delta^{opt}(F',G') = |G - F| < a$ so that loss$(F',G') \geq$loss$(F,G) = c'$. But the inequality (6) yields a loss$(F',G') \leq \tilde{c}(d,a)$ and by definition $\tilde{c}(d,a) < c'$. Therefore there cannot be such F, G. Hence c_z is a loss factor. □

3 Minimality

We now prove that the loss factor from Theorem 1 is minimal.

Theorem 2 *There is no $c < c_z = 1 + 2\frac{d^2}{d+1}$ with: z is a reference point with loss factor c for δ_s, C, and the translations.*

Proof. We give a class of bad cases, by giving one G and a class of figures $\{F_\epsilon\}$ for which

$$\lim_{\epsilon \to 0} \text{loss } (F_\epsilon, G) = 1 + 2\tfrac{d^2}{d+1}$$

Let G be the cone of area 1 and height 1, and F_ϵ its frustrum of height $1-\epsilon$. We make some error-estimation for $\epsilon \to 0$. In contrast to (1) we have to estimate $\delta_s(F_\epsilon, G)$ from below. The inequality in (1) is caused by two sources of error, we call them $b(\epsilon)$ for the periphery of the displacement regions and $h(\epsilon)$ for the overlap of the cap and the displacement region (See Figure 6).

$$\delta_s(F_\epsilon, G) > 2k + |f| - b(\epsilon) - h(\epsilon) \tag{7}$$

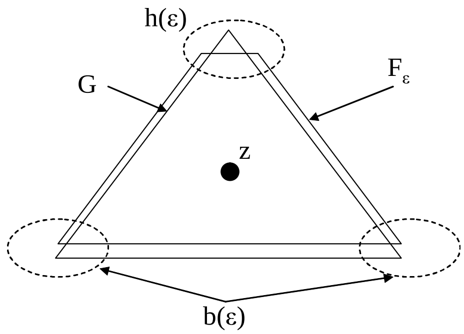

Figure 6: The worst case. The figures are aligned so that the centroids match. The sources of error addressed in the proof are visualized.

The thickness of the displacement region is $O(\epsilon^d)$. therefore $b(\epsilon) = O(\epsilon^{d+1})$ and $h(\epsilon) = O(\epsilon^{2d-1})$. Hence for $d > 1$

$$\delta_s(F_\epsilon, G) > (2k + |f|)(1 + O(\epsilon)) \tag{8}$$

It therefore is sufficient to show $\lim_{\epsilon \to 0} 2k + |f| = c_z$. Straightforward computation gives

$$2k + |f| = 2(\frac{d}{d+1}(1-\epsilon)|f|\frac{d}{1-d|f|} + |f| = (1 + 2(\frac{d^2}{d+1})\frac{1-\epsilon}{1-\epsilon^d})|f| \tag{9}$$

Therefore $\lim_{\epsilon \to 0} 2k + |f| = c_z$.

□

4 Regular Reference Points

It seems interesting to consider a special class of reference points:

Definition 2 *Let T be closed under composition and containing the translations. Then a* regular reference point *with* regular loss factor c *for* δ, Φ, *and the translations is a characteristic point with*

- r *is equivariant for* T.
- *For all* F, G *with* $\delta(F, G) = \delta^{opt}(F, G)$: $\delta(F - r(F), G - r(G)) \leq c\delta^{opt}(F, G)$

A regular reference point is a reference point even for translations. In the definition we use the fact that because of the equivariance one can look solely at figures in optimal matching position. If we have a regular reference point, then translating the optimal match, so that the regular reference points coincide, yields a approximate match. For a general reference point in contrast it is only known, that one approximate solution has coincident reference points.

It seems best to discuss regularity always for the loss factor, not for the reference point itself, because the latter may theoretically be also a general reference point with a smaller loss factor.

Currently no loss factor for a reference point seems to be known which is not at the same time a regular loss factor. It would of course be interesting to have loss factors that are not regular loss factors, i.e. either there is no regular loss factor or it is greater. Artificial examples may be cases, where the transition set is artificially reduced, so that it is not closed under translations and hence the definition of regularity does not apply. However, even this is not trivial.

It is easy to see in the proof of Theorem 1, that the loss factor is regular, because the transformation used in the proof is a translation.

If we have a reference point in translations, then this is already a regular reference point with the same loss factor for all translations, for which it is equivariant. The following theorem was presented e.g. for the centroid in [3].

Theorem 3 *Given a set Φ of figures in R^d, a metric $\delta(F,G) : \Phi \times \Phi \to R$, a set of transformations closed under composition $T \subset (R^d)^{R^d}$ and containing the translations, a* reference point $r : F \to R$ *with* loss factor $c > 1$ *for $\delta, \Phi,$ and the translations . Furtermore r be equivariant for T. Then r is a regular reference point with regular loss factor c for $\delta, \Phi,$ and the translations.*

Proof.
Be $t^{opt} \in T$ a transformation, which brings G in optimal position, i.e. $\delta(F, t^{opt}(G)) = \delta_T^{opt}(F,G)$.
Be $G' = t^{opt}(G)$, Be $t = r(F) - r(G')$, hence $t(r(G')) = r(F)$.
Be $t^o(f) := t(t^{opt}(f))$, $t^o \in T$ because of closedness.
We have to prove: t^o is a approximate match, i.e.$r(t^o(G)) = r(F)$ and $\delta(F, t^o(G)) \leq c\delta_T^{opt}(F,G)$
We have $t^o(G) = t(t^{opt}(G)) = t(G')$
With equivariance we have: $r(t^o(G)) = r(t(G')) = t(r(G')) = r(F)$.
Because r is reference point for translations:

$$\delta(F, t^o(G)) = \delta(F, t(G')) = c \leq \delta(F, G') = c\delta_T^{opt}(F,G)$$

□

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