Quantum Information and Complexity

T Hida
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editors

Proceedings of the Meijo Winter School 2003

Quantum Information and Complexity

Meijo University, Nagoya, Japan
6–10 January 2003

editors

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TRANSCENDING THE LIMITS OF TURING COMPUTABILITY *

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Hypercomputation or super-Turing computation is a "computation" that transcends the limit imposed by Turing's model of computability. The field still faces some basic questions, technical (can we mathematically and/or physically build a hypercomputer?), cognitive (can hypercomputers realize the AI dream?), philosophical (is thinking more than computing?). The aim of this paper is to address the question: can we mathematically build a hypercomputer? We will discuss the solutions of the Infinite Merchant Problem, a decision problem equivalent to the Halting Problem, based on results obtained in Refs. 9 and 2. The accent will be on the new computational technique and results rather than formal proofs.

*VAA is grateful for partial support from the USA Civil Research and Development Foundation (CRDF grant UM1-2090); CSC and BSP have been partially supported by the Auckland University Vice-Chancellor's Development Fund 23124; BSP was also partially supported by JSPS(102704) and RFFI(01149) grants.

1. Introduction

Hypercomputation or super-Turing computation is a "computation" that transcends the limit imposed by Turing's model of computability; for a recent perspective see the special issue of the journal *Minds and Machines* (12, 4, 2002). Currently there are various proposals to break Turing' Barrier by showing that certain classes of *computing procedures* have super-Turing power (see Refs. 19,32,6,7,31,11 and 12). A specific class of computing procedures [15,9,22,2] make essential use of some physical theory, relativity theory in Ref.15, quantum theory, Refs. 9 and 22; they all reflect an attitude advocated by Landauer [23,24] (*information is inevitably physical*) and Deutsch [13,14] (*the reason why we find it possible to construct, say, electronic calculators, and indeed why we can perform mental arithmetic ... is that the laws of physics "happen" to permit the existence of physical models for the operations of arithmetic*).

The aim of the present paper is to revisit the solutions offered in Refs. 9 and 2. We will focus on the novelty of the approach and we will discuss its power and its limits. No proofs will be offered.

2. The Classical Merchant Problem

Recall that in the classical version of the Merchant Problem we have 10 stacks of coins, each stack containing 100 coins, and we know that *at most one stack contains only false coins, weighting* 1.01 g; true coins weight 1 g. The problem is to find the stack with false coins (if any) *by only one weighting*. The classical solution reduces the problem to the weighting of a special combination of coins: one coin from the first stack, two coins from the second stack, ... , ten coins from the tenth stack. If the false coins are present in the N-th stack, then the weight of the combination will be $55 + \frac{N}{100}\,g$; otherwise the weight is just $55\,g$. The Merchant Problem quoted above was widely spread in allies armies during the Second World War, cf. Ref.21. Probably the elegant solution described above was the very first solution of a computational problem bearing typical features of quantum computing, see an extended discussion in Ref.9.

3. The Infinite Merchant Problem

In what follows we are going to consider the following generalization of the problem, the Infinite Merchant Problem: we assume that *we have countable many stacks, given in some computable way, all of them, except at most one, containing true coins only*. True coins weight 1 and false coins weight $1 + 2^{-j}$, $j > 0$. Again we are allowed to take a coin from each stack and we want to determine whether all coins are true or there is a stack of false coins.

Next we will show that the Infinite Merchant Problem is classically undecidable by reducing it to the Halting Problem, i.e. the problem to decide whether an arbitrary Turing machine (TM) halts on an arbitrary input. Assume that a TM operates on positive integers and suppose, for the sake of contradiction, that there exists a TM HALT which can decide whether a TM T (given by its code $\#T$, a positive integer) eventually stops on input x:

$$\mathtt{HALT}(\#T, x) = \begin{cases} 1, & \text{if } T(x) \text{ stops}, \\ 0, & \text{otherwise}. \end{cases}$$

We construct a TM Q

$$Q(x) = \begin{cases} 1, & \text{if } \mathtt{HALT}(x,x) = 0, \\ \text{loops forever}, & \text{otherwise}, \end{cases}$$

and deduce the contradiction:

$$\mathtt{HALT}(\#Q, \#Q) = 1 \text{ iff } \mathtt{HALT}(\#Q, \#Q) = 0.$$

We next describe the reduction. Assume that we have a classical solution of the Infinite Merchant Problem and we are given a TM T and an input x for T. We construct a computable sequence $q_1, q_2, \cdots q_i \cdots$ as follows: if the computation of $T(x)$ did not stop till the i-th step, then we put $q_i = 1$; if the computation halted at step i_0, then we put $q_{i_0} = 1 + 2^{-j}$ and $q_r = 1$, for all $r > i_0$. The sequence q_i satisfies all conditions of the Infinite Merchant Problem and $T(x)$ halts if and only if there is a false coin, i.e., $q_{i_0} = 1 + 2^{-j}$, for some j. This shows that the Infinite Merchant Problem is undecidable as the Halting Problem is undecidable.

In fact the two problems are equivalent. Indeed, assume that we could classically solve the Halting Problem. To every sequence (q_i) satisfying the conditions of the Infinite Merchant Problem we associate the TM T such that $T(i) = 1$ if $q_i = 1$, and $T(i) = 0$ otherwise. The TM T' defined by $T'(0) = \min\{i \mid T(i) = 0\}$ halts at 0 if and only if there an i_0 such that $q_{i_0} = 1 + 2^{-j}$, i.e., $T'(0)$ halts if and only if there are false coins in the system. Hence, a classical solution of the Halting Problem will produce a classical solution for the Infinite Merchant Problem.

The above discussion shows that undecidability is determined by the impossibility to decide in a finite time the answers to an infinite number of questions, "does the first stack contain a false coin?", "does the second stack contain a false coin?", etc. This might be caused either by the fact that the time of the computation grows indefinitely or by the fact that the space of computation grows indefinitely or both. The classical theories of computability and complexity (see, for example, Ref.5) do not give any indication in this respect. In the following section we will show that time can be made finite provided we use a specific probabilistic strategy.

4. A Probabilistic Solution

In this section we present, in a slightly different way, the probabilistic solution proposed in Ref.9. We will adopt the following strategy. We are given a probability $\theta = 2^{-n}$ and we assume that we work with a "device" described below[a] with sensitivity given by a real $\varepsilon = 2^{-m}$. Then, we compute classically a time $T = T_{\theta,\varepsilon}$ and run the "device" on a random input for the time T. If we get a click, then the system has false coins; if we don not get a click, then we conclude that with probability greater than $1-\theta$ all coins are true. An essential part of the method is the requirement that the time limit T is *classically computable.*

The "device" (with sensitivity ε) will distinguish the values of the iterated quadratic form $\langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x}\rangle = \sum_{i=1}^{\infty} q_i^t\, |x_i|^2$, by observing the difference between averaging over trajectories of two discrete random walks with two non-perturbed and perturbed sequences t_l, $\tilde{t}_l$ of "stops". The non-perturbed sequence corresponds to equal steps $\delta_m = 1$, $t_l = \sum_{m=0}^{l} \delta_m$, and the perturbed corresponds to the varying steps Δ_m, $0 < \Delta_m < \delta_m$, $\tilde{t}_l = \sum_{m=0}^{l} \Delta_m$. We work with the intersections of l_2 with the discrete Sobolev class l_2^1 of square-summable sequences with the square norm

$$| \mathbf{x} |_1^2 = \sum_{m=1}^{\infty} |x_m - x_{m-1}|^2, \tag{1}$$

and the discrete Sobolev class $\tilde{l}_2^1$ of weighted-summable sequences with the square norm

$$\| \mathbf{x} \|_1^2 = \sum_{m=1}^{\infty} \frac{1-\Delta_m}{\Delta_m}\, |x_m - x_{m-1}|^2. \tag{2}$$

[a] As in Ref.9 we use quotation marks when referring to our mathematical "device".

By natural extension from cylindrical sets we can define the Wiener measures $\tilde{W}$ and W on the spaces of trajectories of the perturbed and non-perturbed random walks respectively and use the absolute continuity $\tilde{W}$ with respect to W: that is for every W–measurable set Ω,

$$\tilde{W}(\Omega) = \frac{1}{\prod_{l=1}^{\infty} \sqrt{\Delta_l}} \int_{\Omega} e^{-\sum_{m=1}^{\infty} \frac{1-\Delta_m}{\Delta_m} |x_m - x_{m-1}|^2}\, dW.$$

Assume that the "device" revealing the exponential growth of the quadratic form of the iterations $\langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x}\rangle$ clicks if

$$\langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x}\rangle \geq \| \mathbf{x} \|^2 + \varepsilon \| \mathbf{x} \|_1^2 .$$

Thus the "device" sensitivity is defined in terms of the Sobolev norm.

Two cases may appear. If for some $T > 0$, $\langle \mathbf{Q}^T(\mathbf{x}), \mathbf{x}\rangle \geq \| \mathbf{x} \|^2 + \varepsilon \| \mathbf{x} \|_1^2$, then the "device" has clicked and we know for *sure* that there exist false coins in the system. However, it is possible that at some time $T > 0$ the "device" hasn't (yet?) clicked because $\langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x}\rangle < \| \mathbf{x} \|^2 + \varepsilon \| \mathbf{x} \|_1^2$. This may happen because either all coins are true, i.e., $\langle \mathbf{Q}^t(\mathbf{x}), \mathbf{x}\rangle < \| \mathbf{x} \|^2 + \varepsilon \| \mathbf{x} \|_1^2$, for all $t > 0$, or because at time T the growth of $\langle \mathbf{Q}^T(\mathbf{x}), \mathbf{x}\rangle$ hasn't yet reached the threshold $\| \mathbf{x} \|^2 + \varepsilon \| \mathbf{x} \|_1^2$. In the first case the "device" will *never* click, so at each stage t the test-vector $\mathbf{x}$ produces "true" information; we can call $\mathbf{x}$ a "true" vector. In the second case, the test-vector $\mathbf{x}$ is "lying" at time T as we *do* have false coins in the system, but they were not detected at time T; we say that $\mathbf{x}$ produces "false" information at time T.

If we assume that there exist false coins in the system, say at stack j, but the "device" does not click at the moment T, then the test-vector $\mathbf{x}$ belongs to the *indistinguishable set*

$$\mathcal{F}_{\varepsilon,T} = \{\mathbf{x} \in l_2^1 \mid ((1+\gamma)^T - 1) \mid x_j \mid^2 < \varepsilon \| \mathbf{x} \|_1^2, \text{ for some } j\}.$$

In Ref.9 it was proven that the Wiener measure of the indistinguishable set tends to zero as $T \to \infty$:

$$\tilde{W}(\mathcal{F}_{\varepsilon,T}) \leq \left(\frac{\varepsilon}{((1+\gamma)^T - 1 - \varepsilon) \cdot \prod_{m=1}^{\infty} \Delta_m} \right)^{1/2}.$$

This fact is not enough to realize the scheme described in the beginning of this section: we need a more precise result, namely we have to prove that $\tilde{W}(\mathcal{F}_{\varepsilon,T})$ *converges computably to zero.* And, indeed, this is true because:

$$\tilde{W}(\mathcal{F}_{\varepsilon,T}) \leq \eta, \text{ provided } t > \log_{1+\gamma} \left(\frac{\varepsilon}{\eta^2 \prod_{m=1}^{\infty} \Delta_m} + 1 + \varepsilon \right).$$

Denote by $P(\mathcal{N})$ the *a priori* probability of absence of false coins in the system. Then, the *a posteriori probability that the system contains only true coins, when the "device" did not click after running the experiment for the time* T, *is*

$$\mathbf{P}_{\text{non-click}}(\mathcal{N}) > 1 - \frac{1 - P(\mathcal{N})}{P(\mathcal{N})} \cdot \frac{\sqrt{\varepsilon}}{\sqrt{(1+\gamma)^T - 1 - \varepsilon\sqrt{\prod_{m=1}^{\infty} \Delta_m}}}.$$

5. A Brownian Solution Based on Resonance Amplification

In Ref.1 the idea to consider a single act of quantum computation as a scattering process was suggested.

We will first illustrate the method by describing a simple quantum scattering system realizing the quantum C_{NOT} gate, i.e., a quantum gate satisfying exactly to the same truth-table as the classical controlled-NOT gate. The C_{NOT} device has two input and output channels. Each channel can be only in two different states, say $|0\rangle, |1\rangle$. The *in* and *out* states of the control-channel are the same, $|I_{in}\rangle = |I_{out}\rangle$, but the *in* and *out* states of the current-channel may be different, $|J_{in}\rangle \neq |J_{out}\rangle$, depending upon the state $|J_{in}\rangle$ and the control-channel state. The classical controlled-NOT gate has the following truth-table:

I_{in}	J_{in}	I_{out}	J_{out}
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

which describes the effect of the device on the above *in* states,

$\lvert I_{in}\ J_{in}\rangle$	$\longrightarrow$	$\lvert I_{out}\ J_{out}\rangle$
$\lvert 0\ 0\rangle$	$\longrightarrow$	$\lvert 0\ 0\rangle$
$\lvert 0\ 1\rangle$	$\longrightarrow$	$\lvert 0\ 1\rangle$
$\lvert 1\ 0\rangle$	$\longrightarrow$	$\lvert 1\ 1\rangle$
$\lvert 1\ 1\rangle$	$\longrightarrow$	$\lvert 1\ 0\rangle$

The quantum C_{NOT} gate operates not only on the "classical" states $|0\rangle$ and $|1\rangle$, ($C_{\text{NOT}}|ij\rangle = |ik\rangle$, where $i, j \in \{0,1\}$ $k = i \oplus j \pmod 2$), but also on all their linear combinations,

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$\longrightarrow \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|11\rangle + \alpha_{11}|10\rangle.$$

This quantum transformation can be presented via the unitary matrix

$$U = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \\ 0\,0\,1\,0 \end{pmatrix}, \qquad (3)$$

with respect to the canonical basis $(e_1, e_2, e_3, e_4) = (|00\rangle, |01\rangle, |10\rangle, |11\rangle)$. More importantly, C_{NOT} in combination with all 1-qubit gates is *universal* [b] and *it cannot be written as a tensor product of two binary operators* $U = U_1\, U_2,\ U_l \neq I$.

We claim that the matrix U in (3) can be realized as a *scattering matrix* of a special *quantum dot*. First, here is the motivation. Consider Figure 1 in which two isolated quantum wires are placed in proximity and there is a window region in which the two wires are coupled. An electron moving in the window region oscillates between the two quantum wires and the probability of the electron exiting into a specific quantum wire depends on the length of the window. This "switching phenomenon" was discovered by del Alamo and Eugster [3] and intensely discussed in literature, see for instance Refs. 16 and 4. We can arrange the setup in such a way that, under normal conditions, the electron exits from the same wire it enters, but switches to the other wire when a classical extra potential is applied, a realization of the C_{NOT} relay. If the control is quantum too, then we obtain the C_{NOT} gate.

We continue with the mathematical model and assume that the quantum dot is inserted in an one-dimensional quantum wire $\mathbf{R} = (-\infty, \infty) = \mathbf{R}_+ \cup \mathbf{R}_-$ between the wires $\mathbf{R}_+$, $\mathbf{R}_-$ and a single electron may be redirected according to the state of the dot.

We assume also that the *inner* Hamiltonian of the quantum dot is presented by a finite or an infinite diagonal matrix $A = \text{diag}\,\{\alpha_1^2, \alpha_2^2, \dots, \alpha_N^2, \dots\}$ with positive diagonal elements $\alpha_1^2 < \alpha_2^2 < \alpha_3^2 < \dots < \alpha_N^2 < \dots$. We assume that the quantum dot is inserted in an one-

[b] Every classically computable function can be computed by a small universal set of gates like {OR, NOT} or {NAND}. A set of quantum gates $\mathcal{S}$ is called universal if any unitary operation can be approximated with an arbitrary accuracy by a quantum circuit involving gates in $\mathcal{S}$; see more in Refs.17,18,20 and 8.

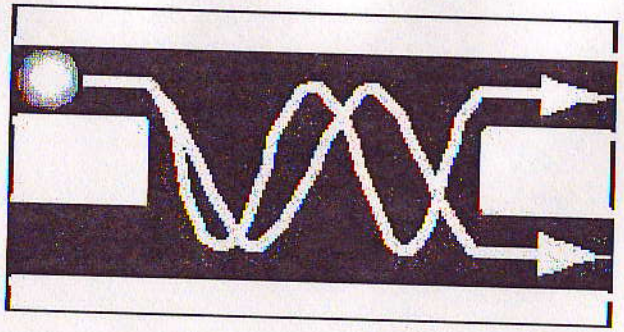

Figure 1. $C_{\rm NOT}$ as quantum dot

dimensional quantum wire $-\infty < x < \infty$ at the origin and a proper boundary condition is satisfied (see (9)) for connecting it with the Schrödinger operator on the wire defined in the space of square-integrable vector-functions $L_2(\mathbf{R}_+, \mathcal{E})$ with values in the infinitely-dimensional Hilbert space $\mathcal{E}$

$$l = -\frac{d^2}{dx^2}\,. \tag{4}$$

One could assign to the above quantum system a product space $H_e \otimes H_d$ constituted respectively by the states of the electron and the states of the dot, and consider an evolution of the system generated by the total Hamiltonian $\mathcal{H}_e + \mathcal{H}_d + \mathcal{H}_{int}$ with a proper interaction term. This would lead to a quite sophisticated problem of quantum mechanics, similar to three-body problems, see for instance Ref.27. We assume now that *the state of the dot is selected independently* and thus reduce the above problem to the corresponding *one-body* problem for an electron scattered in the quantum wire depending on the state of the dot. The corresponding device should be called rather *quantum relay* rather than *quantum gate*; however, it may be transformed into a quantum gate if the state of the dot is obtained as a quantum state with finite life-time. Practically the model suggested below is acceptable if the life-time of the state of the dot is long enough during the scattering experiment.The corresponding general "zero-range" quantum Hamiltonian (solvable model) is described as a self-adjoint extension $\mathbf{A}_\beta$ of the orthogonal sum $l \oplus A$ restricted to $l_0 \oplus A_0$ in $L_2(\mathbf{R}, \mathcal{E}) \oplus E$ onto a proper domain; here $\mathcal{E}$ is the infinitely- dimensional input space and E is the inner space (with dim $(E) \geq 2$). The *spectrum* σ_β of the

operator $\mathbf{A}_\beta$ is absolutely-continuous and fills the positive half-axis $\lambda \geq 0$ with multiplicity dim $(\mathcal{E})$. The role of eigen-functions of the spectral point $p^2 = \lambda > 0$ is played by the *scattered waves* $\overrightarrow{\Psi}_\nu$, $\overleftarrow{\Psi}_\nu$, labeled with vectors $\nu \in \mathcal{E}$. The components of the scattered waves $\Psi_\nu(p)$ in the *outer space* $L_2(\mathbf{R})$ are presented as linear combinations of exponentials:

$$\overrightarrow{\Psi}_{\nu,p}(x) = \begin{cases} e^{-ipx}\nu + e^{+ipx}\overleftarrow{R}(p)\nu, & x < 0, \\ e^{-ipx}\overrightarrow{T}(p)\nu, & x > 0, \end{cases}$$

$$\overleftarrow{\Psi}_{\nu,p}(x) = \begin{cases} e^{ipx}\nu + e^{-ipx}\overrightarrow{R}(p)\nu, & x > 0, \\ e^{ipx}\overleftarrow{T}(p)\nu, & x < 0. \end{cases} \tag{5}$$

The matrix

$$\mathbf{S}_\beta(p) = \begin{pmatrix} \overrightarrow{T}(p) & \overrightarrow{R}(p) \\ \overleftarrow{R}(p) & \overleftarrow{T}(p) \end{pmatrix}, \tag{6}$$

is called the *scattering matrix* of the operator $\mathbf{A}_\beta$.[c]

The evolution of the *wave function* of the quantum mechanical system with Hamiltonian $\mathbf{A}_\beta$ given by the equation

$$\frac{1}{i}\frac{\partial \Psi}{\partial t} = \mathbf{A}_\beta \Psi, \tag{7}$$

and proper initial condition

$$\Psi\Big|_{t=0} = \Psi_0,$$

can be described by the correspopnding evolution operator constructed from the above scattered waves and square-integrable bound states Ψ_s which satisfy the homogeneous equation

$$\mathbf{A}_\beta \Psi_s = \lambda_s \Psi_s,$$

with negative eigen-values λ_s. Bound states do not play an essential role in our construction, so we may assume that the initial state Ψ_0 is orthogonal to all bound states and may be expanded in an analog of Fourier integral over the scattered waves

$$\Psi_0 = \frac{1}{2\pi}\int_R \sum_\nu \Psi_{\nu,p}\langle \Psi_{\nu,p}, \Psi_0\rangle dp.$$

[c]The transmission coefficients appear on the main diagonal of the matrix to fit the physical meaning of the scattering matrix for small values of $|\beta|$, when it is reduced to the undisturbed transmission $\mathbf{S}(p) = I$.

Then the evolution described by the solution of the equation (7) and the above initial data can be presented as a (continuous) linear combination

$$\Psi(t) = \frac{1}{2\pi}\int_{\mathbf{R}}\sum_{\nu} e^{ip^2 t}\Psi_{\nu,p}\langle\Psi_{\nu,p},\Psi_0\rangle dp$$

of *modes* incoming from infinity on the left $(-\infty)$ and on the right $(+\infty)$, and outgoing modes *scattered* to both directions $\pm\infty$ according to the solution of the time-dependent Schrödinger equation $\frac{1}{i}\frac{\partial \mathbf{U}}{\partial t} = \mathbf{AU}$:

$$\mathbf{S}_\beta(p) : e^{ip^2 t}\begin{pmatrix} e^{-ipx}\nu_{left} \\ e^{ipx}\nu_{right}\end{pmatrix} \longrightarrow e^{ip^2 t}\begin{pmatrix} e^{-ipx}(\overrightarrow{T}(p)\nu_{left} & + \overrightarrow{R}(p)\nu_{right}) \\ e^{ipx}(\overleftarrow{R}(p)\nu_{left} & + \overleftarrow{T}(p)\nu_{right})\end{pmatrix}. \tag{8}$$

The analytic structure of the scattering matrix depends upon the structure of the inner Hamiltonian and a sort of interaction between the inner E and outer $L_2(\mathbf{R},\mathcal{E})$ spaces. To avoid the discussion of the general situation we use here a scattering matrix for a solvable model of the quantum dot which is based on zero-range potential with inner structure, introduced in Ref.28 and already used in Ref.4 for the description of typical features of nano-devices, see also Ref.2.

If we choose an infinitely-dimensional input space $\mathcal{E}\oplus\mathcal{E}$ with components corresponding to the amplitudes of the scattered waves at $\pm\infty$ and an one-dimensional *deficiency subspace* $\mathcal{N}_i$ spanned by the normalized vector $e = e_i \in \mathcal{E}$ (see Ref.2) and introduce the scalar function

$$\mathcal{M} = \langle \frac{I+\lambda A}{A-\lambda I}e, e\rangle,$$

then using the interaction defined by the boundary conditions (9) depending on a vector $\beta \in \mathcal{E}$ imposed on the boundary values (the jump $[u'](0)$ and the value $u(0)$ at the origin) of the component of the wave-function in the outer space and the *symplectic coordinates* $\vec{\xi}_\pm \in \mathcal{E}$ of the inner component of the wave-function (see Ref.29):

$$\begin{pmatrix}[u'](0) \\ -\vec{\xi}_-\end{pmatrix} = \begin{pmatrix} 0 & \beta \\ \beta^+ & 0\end{pmatrix}\begin{pmatrix} u(0) \\ \vec{\xi}_+\end{pmatrix}, \tag{9}$$

we obtain the scattering matrix in the form

$$\mathbf{S}_\beta(p) = \begin{pmatrix}\overrightarrow{T}(p) & \overrightarrow{R}(p) \\ \overleftarrow{R}(p) & \overleftarrow{T}(p)\end{pmatrix},$$

with equal transmission and reflection coefficients $\overrightarrow{T}, \overleftarrow{T}, \overrightarrow{R}, \overleftarrow{R}$:

$$\overrightarrow{T}(p) = \overleftarrow{T}(p) = \mathbf{P}_\beta^\perp + \frac{2ip}{2ip + |\beta|^2\mathcal{M}^{-1}}\mathbf{P}_\beta,$$

$$\overrightarrow{R}(p) = \overleftarrow{R}(p) = -\frac{|\beta|^2\mathcal{M}^{-1}}{2ip + |\beta|^2\mathcal{M}^{-1}}\mathbf{P}_\beta.$$

Here $\mathbf{P}_\beta$ is the orthogonal projection of $\mathcal{E}$ onto the one-dimensional space spanned by the vector β, and $\mathbf{P}_\beta^\perp = I - \mathbf{P}_\beta$ is the orthogonal projection on the complimentary space. The constructed solvable model reveals the role of zeroes of the scattering matrix – the *resonances* – in implementing the switching function.

We observe first the behaviour of the scattering matrix at the resonance energy $\alpha_1^2 > 0$ in case the resonance level α_1^2 in the quantum dot is vacant as well as all levels α_l^2 above α_1^2. In this case we have

$$\mathcal{M}_1 = \frac{1+\alpha_1^2\lambda}{\alpha_1^2-\lambda}|e_1|^2 + \sum_{l=2}^{N}\frac{1+\alpha_l^2\lambda}{\alpha_l^2-\lambda}|e_l|^2 = \frac{1+\alpha_1^2\lambda}{\alpha_1^2-\lambda}|e_1|^2 + \mathcal{M}_3,$$

where $|e_l|^2$ are the squares of the Fourier coefficients of the deficiency vector e with respect to the eigen-vectors of the operator A.

Next we consider the case when the resonance level α_1^2 is occupied. In this case

$$\mathcal{M}_2 = \frac{1+\alpha_2^2\lambda}{\alpha_2^2-\lambda}|e_2|^2 + \sum_{l=3}^{N}\frac{1+\alpha_l^2\lambda}{\alpha_l^2-\lambda}|e_l|^2 = \frac{1+\alpha_2^2\lambda}{\alpha_2^2-\lambda}|e_2|^2 + \mathcal{M}_3.$$

In the above analysis we have ignored the electron spin, that is we have assumed that all electrons have the same constant spin on the quantum circuit $\mathbf{R}_- \cup \mathbf{R}_+$, with the quantum dot attached.[d] We may assume that the circuit lies on the surface of a semiconductor with Fermi-level α_1^2 (see Ref.26).

We assume that the state of the dot with the level α_1^2 vacant corresponds to $I_{in} = I_{out} = 0$ and the state of the dot with the level α_1^2 occupied corresponds to $I_{in} = I_{out} = 1$. We identify these states of the system as state S_1 and state S_2, respectively.

[d] Our hypothesis is satisfied in case the travelling electrons are polarized and the electron on the level α_1^2 is polarized. Note that Pauli's exclusion principle is still valid, but with only one electron on each orbital: the magnetic field is absent, so the polarization is not changed during the experiment.

For every vector β the transmission coefficients on the resonance electron's energy $\lambda = \alpha_1^2$ can be expressed as (see Ref.2):

$$\overrightarrow{T}(p) = \overleftarrow{T}(p) = \mathbf{P}_\beta^\perp + \frac{2ip}{2ip + |\beta|^2 \mathcal{M}_1^{-1}} \mathbf{P}_\beta = I. \ \overrightarrow{R}(p) = \overleftarrow{R}(p) = 0,$$

(at the resonance energy we have $\mathcal{M}_1^{-1} = 0$), so the scattering matrix becomes the identity when α_1^2 *is not occupied.*

In the second case, when the resonance level $p^2 = \lambda = \alpha_1^2$ is occupied, we obtain (due again to Pauli's exclusion principle) the following expression for the transmission coefficients of passing electrons with resonance energy:

$$\overrightarrow{T}(p) = \overleftarrow{T}(p) = \frac{2ip\mathcal{M}_2(\lambda)}{2ip\mathcal{M}_2(\lambda) - |\beta|^2}$$

$$= \frac{2ip\left(\frac{1+\alpha_2^2\alpha_1^2}{\alpha_2^2-\alpha_1^2}|e_2|^2 + \sum_{l=3}^N \frac{1+\alpha_l^2\alpha_1^2}{\alpha_l^2-\alpha_1^2}|e_l|^2\right)}{2ip\left(\frac{1+\alpha_2^2\alpha_1^2}{\alpha_2^2-\alpha_1^2}|e_2|^2 + \sum_{l=3}^N \frac{1+\alpha_l^2\alpha_1^2}{\alpha_l^2-\alpha_1^2 a}|e_l|^2\right) + |\beta|^2},$$

and the corresponding expressions for the reflection coefficients

$$\overrightarrow{R}(p) = \overleftarrow{R}(p) = -\frac{|\beta|^2}{2ip\mathcal{M}_2 + |\beta|^2}\mathbf{P}_\beta,$$

which can be approximated, for large enough β, as

$$\overrightarrow{T}(\alpha_1) = \overleftarrow{T}(\alpha_1) \approx \mathbf{P}_\beta^\perp, \ \overrightarrow{R}(\alpha_1) = \overleftarrow{R}(\alpha_1) \approx -\mathbf{P}_\beta.$$

Hence the scattering matrix is equal to

$$\mathbf{S}_\beta(\alpha_1^2) = \begin{pmatrix} \mathbf{P}_\beta^\perp & -\mathbf{P}_\beta \\ -\mathbf{P}_\beta & \mathbf{P}_\beta^\perp \end{pmatrix} = I - \begin{pmatrix} \mathbf{P}_\beta & \mathbf{P}_\beta \\ \mathbf{P}_\beta & \mathbf{P}_\beta \end{pmatrix}, \tag{10}$$

for relatively large enough β, if the resonance level α_1^2 *is occupied.*

We continue by showing how the probabilistic approach discussed in the previous section can be realised.We consider now an imaginable quantum scattering system with an infinitely-dimensional input-space and, in particular, with the infinite dimensional space $\mathcal{E}$ for which a solvable model was described above as an extension of the orthogonal sum $l_0 \oplus A_0$ with boundary condition (9), in which , however, the β-channel connecting the outer subspace $L_2(\mathbf{R}, \mathcal{E})$ with the inner subspace E is as before two- dimensional. We associate these extensions with two states (S_1, S_2) of the total quantum system combined of the inner and outer components, with the interaction respectively *switched on* via the boundary condition (9), $\beta \neq 0$, or *switched*

off, $\beta = 0$, and we interpret the Halting Problem in a probabilistic setting as the problem of distinguishing of the states (S_1, S_2) of the quantum system via a scattering experiment with a random input.

Following the probabilistic strategy in Section 4, we compare the scattering matrices $\mathbf{S}_\beta(\alpha_1^2)$ in states S_1 and S_2. In the first state this matrix coincides with identity, hence

$$\frac{1}{2}(I + \mathbf{S}_\beta(\alpha_1^2)) = I.$$

In the second state we have

$$\frac{1}{2}(I + \mathbf{S}_\beta(\alpha_1^2)) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} \mathbf{P}_\beta & 0 \\ 0 & \mathbf{P}_\beta \end{pmatrix} \mathcal{P}_{sym},$$

where $\mathcal{P}_{sym} = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$ is the projection onto the symmetric subspace of the input space $\mathcal{E} \oplus \mathcal{E}$ consisting of vectors with equal componenets $\begin{pmatrix} e \\ e \end{pmatrix}$. The projections $\begin{pmatrix} \mathbf{P}_\beta & 0 \\ 0 & \mathbf{P}_\beta \end{pmatrix}$ and $\mathcal{P}_{sym}$ commute and their product gives the projection onto the space spanned by $\overrightarrow{\beta} = \begin{pmatrix} \beta \\ \beta \end{pmatrix}$. Hence,

$$\frac{1}{2}(I + \mathbf{S}_\beta(\alpha_1^2)) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \mathcal{P}_\beta,$$

where $\mathcal{P}_\beta$ is the orthogonal projection on the subspace $\mathcal{E}$ which is collinear to the vector $\overrightarrow{\beta}$ in the input space. Therefore, in the second case, for every vector $e \in \mathcal{E} \oplus \mathcal{E}$ we have:

$$\frac{1}{2}(\langle e, e\rangle + \langle \mathbf{S}_\beta(\alpha_1^2)e, e\rangle) = |e|^2 - |\mathcal{P}_\beta e|^2.$$

The expectation is that if the probability of the event $\mathcal{P}_\beta e = 0$ is zero, then by choosing a random test-vector, with probability 1 the above correlation is strictly less than 1. To obtain the corresponding quantitative result we will assume that we have a testing "device" distinguishing between the two states of the system , which "clicks" if

$$|\mathcal{P}_\beta e|^2 > \varepsilon |e|^2. \tag{11}$$

Unfortunately, the above "device" is not sensitive enough to derive proper estimates for probabilities and we need another norm in the right-hand side of the last inequality. In our case the input space $\mathcal{E}$ is l_2 with the standard orthogonal basis $\mathbf{x} \longrightarrow \{x_m\}_{m=0}^\infty$. Following Ref.9 we consider the discrete Sobolev classes and norms introduced in Section 4, (1) and (2) in order to define the case when the "device" clicks. Next we assume that the (complex) increments $x_m - x_{m-1}$ are independent. We are going to use, together with l_2 two more spaces of test-vectors. Both are stochastic spaces of all trajectories $\mathbf{x}(t)$ of a Brownian particle on the complex plane along different discrete sequences of intermediate moments of time ("stops"): the equidistant sequence $t_l = \sum_{m=1}^{l} \delta_m$, for the first space, and the perturbed sequence $\tilde{t}_l = \sum_{m=1}^{l} \Delta_m$, for the second space. Both spaces are equipped with proper Wiener measures W, $\tilde{W}$ (see Ref.30). The measure W is defined on the algebra of all finite-dimensional cylindrical sets $C^{t_1,t_2,\dots,t_N}_{\Delta_1,\Delta_2,\dots,\Delta_N}$ of trajectories with fixed initial point $x_0 = 0$ and "gates" Δ_l, $l = 1, \dots, N$ (which are open discs in the complex plane):

$$C^{t_1,t_2,\dots,t_N}_{\Delta_1,\Delta_2,\dots,\Delta_N} = \{\mathbf{x} \mid x_{t_l} \in \Delta_l, l = 1,2,\dots,N\},$$

via multiple convolutions of the Green functions $G(x_{l+1}, t_{l+1} | x_l, t_l)$ corresponding to the sequence $\delta_{l+1} = t_{l+1} - t_l$:

$$W(C^{t_1,t_2,\dots,t_N}_{\Delta_1,\Delta_2,\dots,\Delta_N}) =$$

$$\frac{\int \cdots \int_{\Delta_N,\Delta_{N-1},\dots,\Delta_1} \frac{dx_1 dx_2 \dots dx_N}{\pi^{\frac{N}{2}} \sqrt{\delta_N \delta_{N-1} \cdots \delta_1}} e^{-\frac{|x_N - x_{N-1}|^2}{\delta_N}} \cdots e^{-\frac{|x_1 - x_0|^2}{\delta_1}}}{\int \cdots \int_{\mathbf{R}_N,\mathbf{R}_{N-1},\dots,\mathbf{R}_1} \frac{dx_1 dx_2 \dots dx_N}{\pi^{\frac{N}{2}} \sqrt{\delta_N \delta_{N-1} \cdots \delta_1}} e^{-\frac{|x_N - x_{N-1}|^2}{\delta_N}} \cdots e^{-\frac{|x_1 - x_0|^2}{\delta_1}}}, \tag{12}$$

where $\mathbf{R}_N = \mathbf{R}_{N-1} = \cdots = \mathbf{R}_1 = \mathbf{R}$. Using the convolution formula, the denominator of (12) can be reduced to the Green function $G(x_N, t_N \mid 0, 0)$, for any $\tau \in (s, t)$:

$$G(x, t \mid y, s) = \int_{-\infty}^{\infty} G(x, t \mid \xi, \tau) G(\xi, \tau \mid y, s) d\xi.$$

In a similar way we can define the Wiener measure for trajectories corresponding to the "perturbed" sequence $\tilde{t}_l$.

In what follows we are going to use the *absolute continuity* of the *perturbed* Wiener measure $\tilde{W}$ with respect to the non-perturbed one W: for every W–measurable set Ω,

$$\tilde{W}(\Omega) = \frac{1}{\prod_{l=1}^{\infty} \sqrt{\Delta_l}} \int_{\Omega} e^{-\sum_{m=1}^{\infty} \frac{1-\Delta_m}{\Delta_m} |x_m - x_{m-1}|^2} dW. \tag{13}$$

Further we consider the class of *quasi-loops*, that is the class of all trajectories of the "perturbed process" which begin from $(x_0, \tilde{t}_0) = (0, 0)$ and for any t, $\max_{0<s<t} |x_s|^2 < Ct$. We note that

- every $\mathbf{x} \in l_2^1$ is a quasi-loop (with $C = \mid \mathbf{x} \mid_1^2$),
- due to the reflection principle (see Ref.30, p. 221), the class of all quasi-loops has Wiener measure one, both in respect of W, $\tilde{W}$.

We assume that the device clics, if the result of averaging exceeds a certain level defined by the above norm $\| \cdot \|_1^2$:

$$|\mathcal{P}_\beta \mathbf{x}|^2 > \ \varepsilon \parallel \mathbf{x} \parallel_1^2 .$$

This device cannot identify the state of the system from the observation of the Breit-Wigner averaged correlation between the input and output of a single act of scattering when presented a randomly chosen input $\mathbf{x} \in \mathcal{E}$ if $|\mathcal{P}_\beta \mathbf{x}|^2 < \varepsilon \parallel \mathbf{x} \parallel_1^2$. This means that the test-vector $\mathbf{x}$ belongs to the *indistinguishable set*

$$\mathcal{F}_\varepsilon = \left\{ \mathbf{x} \in l_2 \cap l_2^1 ,\, |\mathcal{P}_\beta \mathbf{x}|^2 < \ \varepsilon \left(\sum_{m=1}^{\infty} \frac{1 - \Delta_m}{\Delta_m} |x_m - x_{m-1}|^2 \right) \right\} \tag{14}$$

$$= \{\mathbf{x} \in l_2 \cap l_2^1 ,\, |\mathcal{P}_\beta \mathbf{x}|^2 < \ \varepsilon \parallel \mathbf{x} \parallel_1^2\}.$$

Though technically we may easily consider, with Breit-Wiegner averaging, the iterated scattering processes described by the powers $\mathbf{S}^m$ of the scattering matrix, we will analyze now the independent single acts of scattering. In this case the indistinguishable set depends only upon the positive number ε, the vector $\vec{\beta} \in \mathcal{E} \oplus \mathcal{E}$ defining the interaction in the quantum system, and the sequence Δ. Without loss of generality we may assume that the vector $\vec{\beta}$ has all non-zero components $\vec{\beta}_l \neq 0$. We assume that the vector $\mathbf{b} = \{b_l\}_{l=1}^{\infty}$, $b_l = \sum_{m=l}^{\infty} \vec{\beta}_m$ belongs to l_2:

$$|b|_l^2 = \sum_{m=1}^{\infty} m^2 |\vec{\beta}_m|^2 < \infty. \tag{15}$$

Our main result reads: *If the condition (15) is satisfied, then the Wiener probability $\tilde{W}(\mathcal{F}_{\varepsilon,1})$ of the indistinguishable set $\mathcal{F}_{\varepsilon,1}$ corresponding to a single act of scattering is finite and is estimated as*

$$\tilde{W}(\mathcal{F}_\varepsilon) < \frac{\sqrt{\varepsilon} |\vec{\beta}|}{\prod_{l=1}^{\infty} \sqrt{\Delta_l} \sqrt{\varepsilon |\vec{\beta}|^2 + |b|^2}}. \tag{16}$$

Following the calculation presented in Ref.9, we approximate the indistinguishable set with finite-dimensional cylinder sets and reduce the estimation of $\tilde{W}(\mathcal{F}_{\varepsilon,1})$ to the calculation of a Wiener integral with respect to the W measure on trajectories associated with "equidistant stops". We have:

$$\tilde{W}(\mathcal{F}_{\varepsilon,1})$$

$$\leq \int_{\text{quasi-loops}, x_0=0\, |\langle \mathbf{x}, \vec{\beta}\rangle|^2 < \varepsilon \|\mathbf{x}\|_1^2} d\tilde{W}$$

$$= \lim_{C\to\infty} \int_{\sup_{s\leq l} |x_s| < C\sqrt{l}, l \leq N,\, x_0=0 |\langle \mathbf{x}, \vec{\beta}\rangle|^2 < \varepsilon \|\mathbf{x}\|_1^2} d\tilde{W}$$

$$= \frac{1}{\prod_{l=1}^{\infty}\sqrt{\Delta_l}} \lim_{C\to\infty}\lim_{N\to\infty}$$

$$\frac{\int_{\sup_{s\leq l}|x_s|<C\sqrt{l}, l\leq N,\, x_0=0} \int\int\cdots\int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_N \prod_{j=1}^{N} e^{-\frac{|x_j - x_{j-1}|^2}{\Delta_j}}}{\pi^N \int_{|x_N|<C\sqrt{N}} dx_N e^{-\frac{x_N - x_0}{t_N}}}$$

$$\leq \frac{1}{\prod_{l=1}^{\infty}\sqrt{\Delta_l}} \lim_{C\to\infty}\lim_{N\to\infty}$$

$$\frac{\int_{\sup_{s\leq l}|x_s|<C\sqrt{l}, l\leq N,\, x_0=0} \int\int\cdots\int_{-\infty}^{\infty} dx_1 dx_2 \dots dx_N e^{-\sum_{s=1}^{N}\frac{|x_s - x_{s-1}|^2}{\delta_s}} e^{\mathcal{B}}}{\pi^N \int_{|x_N|<C\sqrt{N}} dx_N e^{-\frac{|x_N - x_0|^2}{t_N}}}.$$

The integrand of the inner integral in the numerator contains the exponential factor

$$e^{\mathcal{B}} = e^{-(1-\Delta_N)\frac{|x_N - x_{N-1}|^2}{\Delta_N} - (1-\Delta_{N-1})\frac{|x_{N-1} - x_{N-2}|^2}{\Delta_{N-1}} - (1-\Delta_1)\frac{|x_1 - x_0|^2}{\Delta_1}},$$

which can be estimated due to (14) by the exponential:

$$e^{-\frac{1}{\varepsilon}|\mathcal{P}_{\vec{\beta}}\mathbf{x}|^2} = e^{-\frac{1}{\varepsilon|\vec{\beta}|^2}|\langle \mathbf{x}, \vec{\beta}\rangle|^2}.$$

Using this equality, the exponential in the numerator can be estimated from below by the quadratic form

$$-\sum_{m=1}^{N}|x_m - x_{m-1}|^2 - \frac{1}{\varepsilon|\vec{\beta}|^2}|\langle \mathbf{x}, \vec{\beta}\rangle|^2. \qquad (17)$$

This quadratic form can be simplified using new vector variables $\xi_m = x_m - x_{m-1}$:

$$\langle \vec{\beta}, \mathbf{x}\rangle = \sum_{m=1}^{\infty} x_m \vec{\beta}_m = \sum_{m=1}^{\infty} \xi_m \overline{\sum_{l=m}^{\infty}\vec{\beta}_l}.$$

Recall that the vector $\mathbf{b}$, $b_m = \sum_{l=m}^{\infty}\beta_l$ belongs to l_2. Then the quadratic form in the exponent of the numerator can be presented as a quadratic form of an operator

$$\langle \xi, A_\varepsilon \xi\rangle = |\xi|^2 + \frac{1}{|\vec{\beta}|^2\varepsilon}|\langle \mathbf{b}, \xi\rangle| = \langle \xi, \left(\mathbf{I} + \frac{|\mathbf{b}|^2}{|\vec{\beta}|^2\varepsilon}\mathbf{P}_{\mathbf{b}}\right)\xi\rangle,$$

where $\mathbf{P}_b$ is the orthogonal projection onto the one-dimensional subspace in l_2 spanned by the vector $\mathbf{b}$. The ratio of the N-dimensional Gaussian integral in the numerator, normalized by the factor $\pi^{-N/2}$ and the Gaussian integral in the denominator can be expressed as

$$\frac{1}{\pi^{N/2}}\int\int\cdots\int e^{-\langle \xi, A_\varepsilon \xi\rangle} d\xi_1 d\xi_2 \dots d\xi_N = \frac{1}{\sqrt{\det A_\varepsilon}} = \frac{\sqrt{\varepsilon}\|\vec{\beta}\|}{\sqrt{\varepsilon|\vec{\beta}|^2 + |b|^2}}.$$

Finally, we obtain the announced result by taking into account the omitted factor $\prod_l \sqrt{\Delta_l}$.

Acknowledgement

We thank Radu Ionicioiu for his comments on a draft form of this paper. Calude and Pavlov have been supported in part by The Vice-Chancellor's University Development Fund 23124/2002.

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