The Collatz Problem[*], the Halting Problem and Randomness

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When he was a student L. Collatz posed the following problem: given any integer a_1 there exists a natural N such that $a_N = 1$, where

$$a_n = \begin{cases} a_n/2, & \text{if } a_n \text{ is even,} \\ 3a_n + 1, & \text{otherwise}. \end{cases}$$

There is a huge literature on this problem and various natural generalisations: see [6, 5, 3, 7]. P. Erdös has said (cf. [6]) that

Mathematics may not be ready for such problems.

A problem/conjecture is *finitely refutable* if verifying a finite number of instances suffices to disprove it. A systematic enumeration (of the problem's search domain) will find a counter-example if one exists. If the search stops, the conjecture is false; if the search does not halt the conjecture is true. For a finitely refutable problem Π we can construct a program C_Π such that

$$\Pi \text{ is false iff } C_\Pi \text{ halts.}$$

For example, $\Pi =$ the Riemann hypothesis is finitely refutable and one can construct a program C_Π of 7,780 bits, cf. [2].

The twin prime conjecture

$$\forall n \, \{\exists p \, [p > n \,\&\, p \text{ prime} \,\&\, p + 2 \text{ prime}]\}$$

is not finitely refutable but can still be solved by testing the halting status of a small program. The stronger twin prime conjecture is finitely refutable:

$$\forall n \, \{\exists p \, [n < p < 2^{n+4} \,\&\, p \text{ prime} \,\&\, p + 2 \text{ prime}]\}.$$

In [2] it was proved, in a non-constructive manner, that Collatz problem is finitely refutable. The non-constructive proof shows that there exists a program C_{Collatz} such that

[*] Also called Syracuse conjecture, the $3x + 1$ problem, Kakutani's problem, Hasse algorithm, and Ulam's problem. See [9].

$$C_{\text{Collatz}} \text{ is false iff } C_{\text{Collatz}} \text{ halts.}$$

Question 1. Can you find/write such a program?

The initial iterates of Collatz map $f(n) = a_n$ exhibit a 'random' character, cf [4]. More precisely, the initial iterates of a randomly selected integer appear to be even or odd with equal probability. Such a result can be rigorously justied if one takes the interval $1 \leq n \leq 2^k$ and considers only the first k iterations (see [6] Theorem A]).

Question 2. Further study randomness properties of the Collatz map.

References

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[9] Collatz Conjecture.